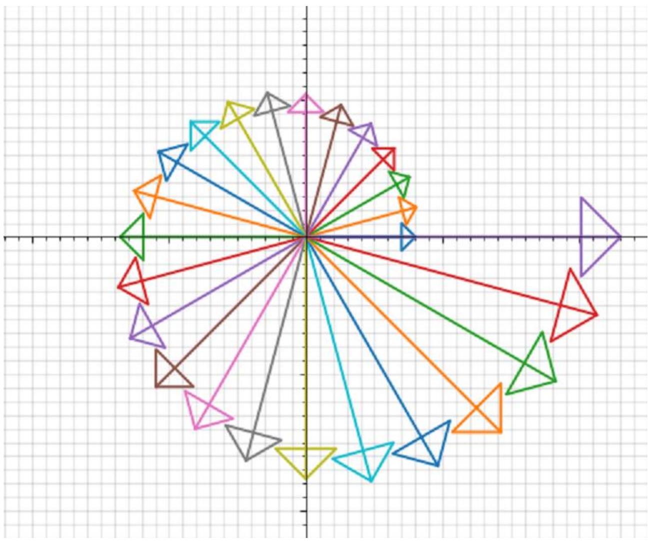

POLAR COORDINATES WITH PYTHON: VOLUME ONE

XY and Polar Coordinate Basics

ABSTRACT

Volume one (of two) reviews the Math basics related to XY and Polar Coordinates

Armand Laurino

XY and Polar Coordinates

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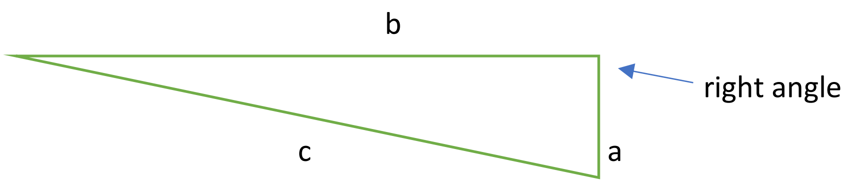
Math related Background topics

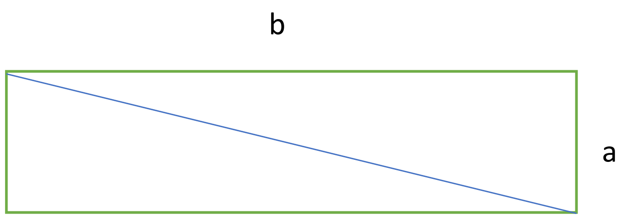
The Right Triangle

Here is your basic right triangle.



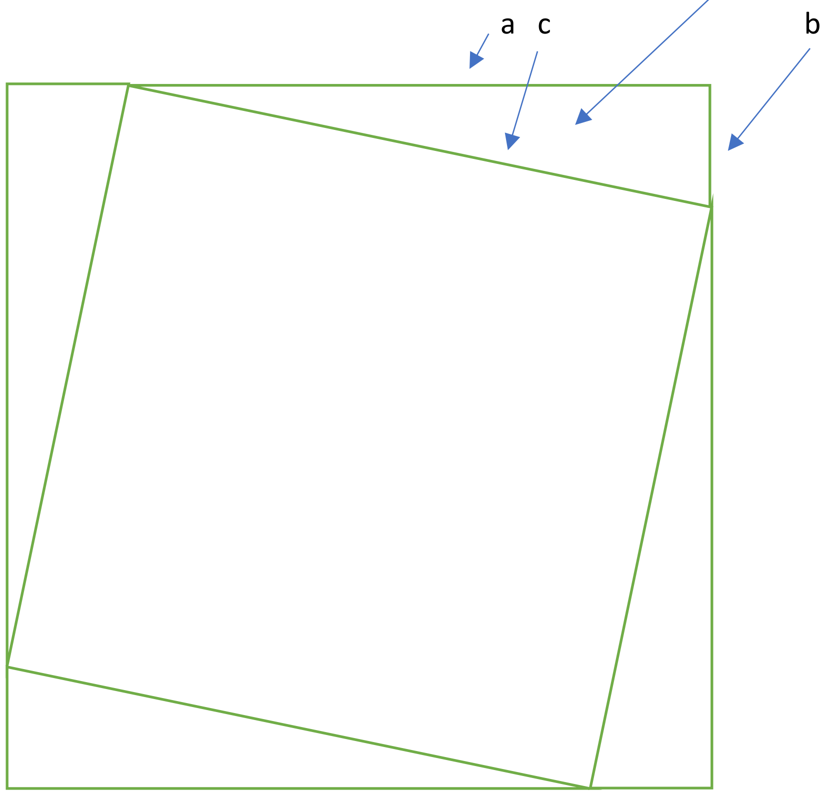
The two sides with lengths a and b are the right triangle's **legs**. The side opposite the right angle (length c) is called the '**hypotenuse'** - Latin *hypotēnūsa,* borrowed from Greek *hypoteínousa* "stretching under" (the right angle)

You can construct a rectangle with side a and side b. The rectangle's diagonal is c.



The area of the rectangle is (a * b)
The area of the two congruent triangles is (a * b)/2 (half of the rectangle)

Here is a construction with **two squares** one inside the other. We used 4 copies of our example right triangle from above. Convince yourself that the inner quadrilateral (4 sides) is indeed a **square**! Hint: the three angles of a triangle add up to 180 degrees, e.g., our right triangle **may** have the angles 90, 70 and 20 degrees.



The outer (larger) square has side a+b and area **(a+b)*(a+b)**
The inner square has side c and area **c*c**.
Each of the four triangles has area **(a*b)/2**

The **outer square** area equals the area of the **inner square** plus the areas of the **4 triangles**!.
So, with a little elementary algebra we see…

(a+b)*(a+b) = **c*c** + **4 * (a*b)/2**
a*a + 2a*b + b*b = c*c + 2a*b
a*a + b*b = c*c
$a^2 + b^2 = c^2$

The above line is the famous Pythagorean Theorem:

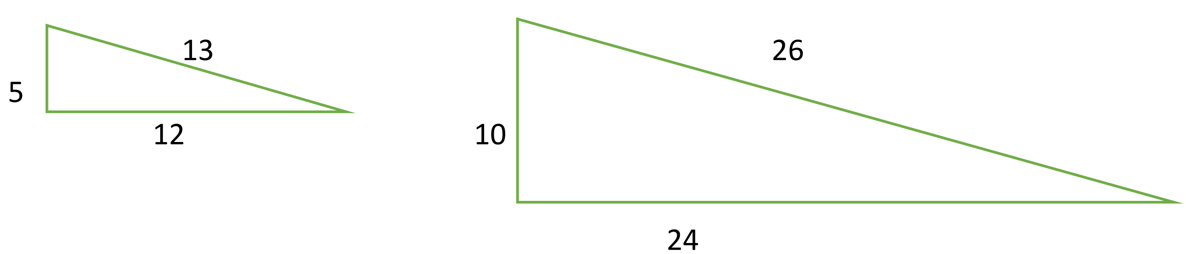
“In a right triangle, the sum of the squares of the legs is equal to the square of the hypotenuse”

Similar Triangles

In simple layman’s terms two triangles with the same shape are called ‘similar.’ Similar triangles can be assorted sizes but must have the same shape.

Take your favorite photo and have it enlarged – say three times bigger! We now have two photos the original and the enlargement. If the enlargement is done correctly a single object in the photos will have the identical shape in the two photos. The shape does NOT change due to the enlargement.

Here are two similar right triangles



We use the Pythagorean theorem to verify that we have valid dimensions for right triangles

$5^2 + 12^2 = 13^2$	$10^2 + 24^2 = 26^2$
25 + 144 = 169	100 + 576 = 676
169 = 169	676 = 676

Our larger triangle is twice the size as the smaller Here, our ‘enlargement’ is two times bigger.

The triangle’s sides are in proportion. This can be stated or expressed in many ways.

5*2 = 10   12*2 = 24   13*2 = 26
5:10::12:24::13:26
5/10 = 12/24 = 13/ 26 = **1/2** ratio

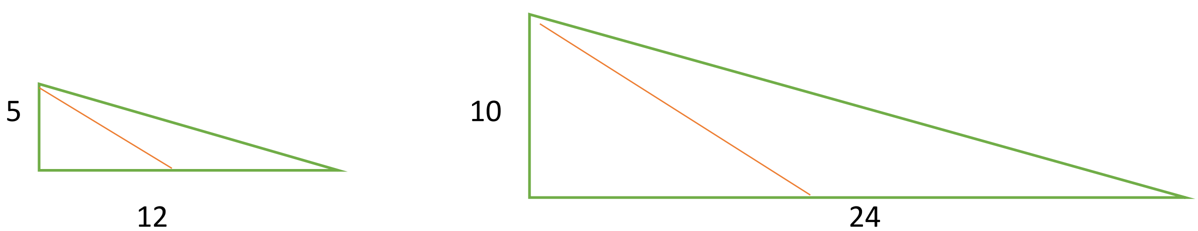
Note: A ratio is a way of comparing two numbers by division (A/B where B is NOT zero!).

In geometry the numbers A and B are usually positive numbers as they are measurements. In Trigonometry we will see zero (for A only) and negative values also!

There are other ways - besides a ratio - to compare **two** numbers. For example, we can say the **25** is 20 more than **5**.

Back to ratios. **25** is five times bigger than **5** - using a ratio we see **5/25 = 1/5** OR **25/5 = 5/1**

Here are our two similar triangles. Anything related to the following two triangles that can be measured with a ruler will obey the **1/2** ratio.



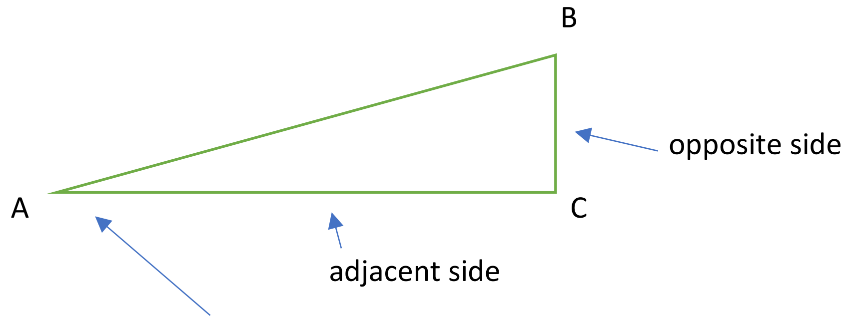
For an example we construct two **orange** line segments that bisect the longer leg of our triangles. We have created two ***new*** right triangles with legs 5, 6 and 10, 12 respectively!

Each line segment is ***new*** triangle(s) hypotenuse. Applying Pythagorean theorem, we see:

Small Triangle	***Large Triangle (ratio 1:2)***	***In General (for ratio 1:N)***
$5^2 + 6^2 = c^2$	$10^2 + 12^2 = c^2$	$(N*5)^2 + (N*6)^2 = c^2$
$25 + 36 = c^2$	$100 + 144 = c^2$	$N^2 (5^2 + 6^2) = c^2$
$61 = c^2$	$244 = c^2$	$N^2 (25 + 36) = c^2$
	$4* 61 = c^2$	$N^2 * 61 = c^2$
	√(4*61) = c	√(N² * 61) = c
	√4* √61 = c	√N2 * √61 = c
√61 = c	2 * √61 = c	N * √61 = c

The Greeks were (still are) fascinated by ratios and proportional shapes. Geometry, architecture, and all forms of art are examples where ratios and proportionality come into play!

Comparing the sides of a single right triangle with vertex points A,B and C.



We consider the acute ∠BAC. Using this angle as a reference we name the side BC the 'opposite' side. We name side AC as the 'adjacent' side. Side AB is named the hypotenuse. There is a total of six related ratios; each has a name - the names are unfortunately confusing! Notice the reciprocal relationships!

sine(∠BAC)	= opposite / hypotenuse	BC / AB
cosine(∠BAC)	= adjacent / hypotenuse	AC / AB
tangent(∠BAC)	= opposite / adjacent	BC / AC
secant(∠BAC)	= hypotenuse /adjacent	AB / AC
cosecant(∠BAC)	= hypotenuse / opposite	AB / BC
cotangent(∠BAC)	= adjacent / opposite	AC / BC

We can (and do!) apply the Pythagorean theorem and perform some basic algebra

$\text{hypotenuse}^2 = \text{opposite}^2 + \text{adjacent}^2$

$1 = (\text{opposite}^2 + \text{adjacent}^2) / \text{hypotenuse}^2$

$1 = \text{opposite}^2 / \text{hypotenuse}^2 + \text{adjacent}^2 / \text{hypotenuse}^2$

$1 = (\text{opposite}/ \text{hypotenuse})^2 + (\text{adjacent} / \text{hypotenuse})^2$

$1 = (\text{sine}(\angle BAC))^2 + (\text{cosine}(\angle BAC))^2$

When we switch and use the acute ∠ABC as the reference the ratios are

sine(∠ABC)	= opposite / hypotenuse	AC / AB
cosine(∠ABC)	= adjacent / hypotenuse	BC / AB
tangent(∠ABC)	= opposite / adjacent	AC / BC
secant(∠ABC)	= hypotenuse /adjacent	AB / BC
cosecant(∠ABC)	= hypotenuse / opposite	AB / AC
cotangent(∠ABC)	= adjacent / opposite	BC / AC

Remember, we are dealing exclusively with right triangles. Here are two of the many relationships.

1. ∠BAC + ∠ABC = 90 degrees
2. sine(∠ABC) = cosine(∠BAC)
3. $1 = (\text{sine}(\angle BAC))^2 + (\text{cosine}(\angle BAC))^2$

Other similar shapes

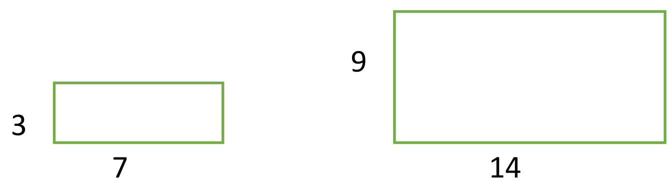
We have been discussing the concept of similarity by using triangles as our geometric object. The concept of similarity is NOT limited to triangles. Examples and notes follow.

All squares are similar.

Two rectangles may or may not be similar.

These two rectangles are similar.

These two rectangles are not similar.

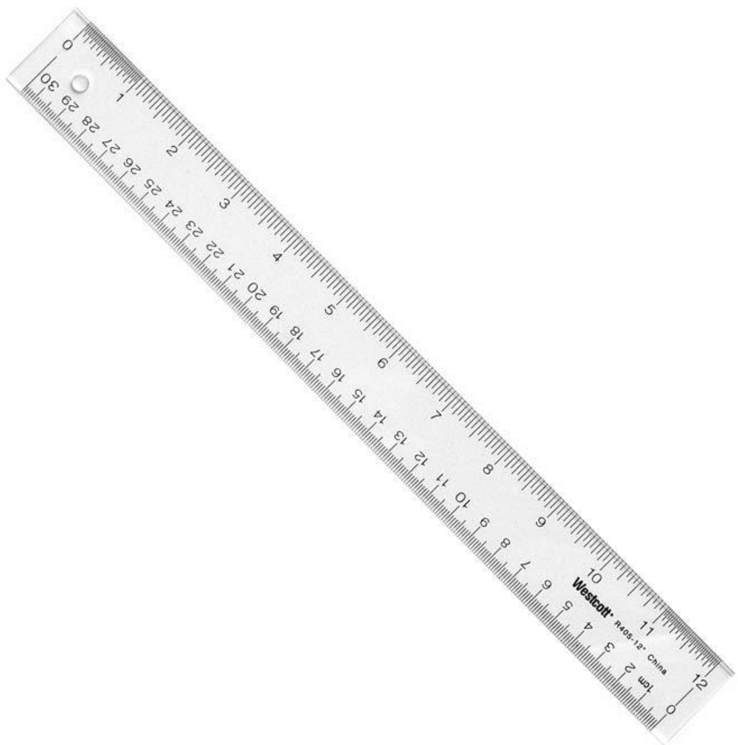


A cool observation is that all circles are similar. They all have the same shape – more about circles a bit later.

Earlier we looked at our pair of similar triangles where one was double (1/2 ratio) in size; we said:

> "Anything related to our two (similar)triangles that can be measured with a ***ruler*** will obey the **1/2** ratio."

Instead of a ***ruler*** we can use a ***flexible tape measure***. The tape measure can be used to measure line segments and curved shapes – like a circular window in our enlarged photo example! The important thing is that here we are measuring in linear units – e.g., the length of a rectangle or the circumference of a circle - versus NON-linear measurements like the area of a rectangle.

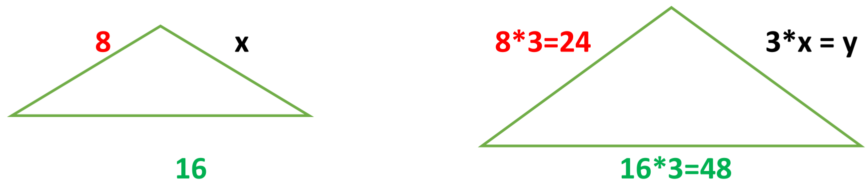


Example, given two rectangles with lengths 3,7 and 9, 21, respectively. The two rectangles are similar with the ratio **1/3**.

A linear measurement : The perimeter of a rectangle is an example of a linear measurement. Here, the two perimeters are 3+3+7+7 = 20 and 9+9+21+21=60 **where 20/60 = 1/3**

A non-linear measurement: The unit of measurement for areas is squares – as in '21 square feet' – think of buying a carpet vs. a fence (perimeter). The two rectangle areas are 3*7=21 sq. units and 9*21 = 189 sq. units **where 21/189 = 1/9 ≠ 1/3.** BUT!! note that **1/9 =** $\mathbf{(1/3)^2}$ **-** cool if you are a Math nerd.

Note the various combinations we can use to compare linear traits of two similar shapes. Suppose these two triangles are similar.



Using ratios to compare numbers

We can compare the sides of a **single (one)** triangle like this.

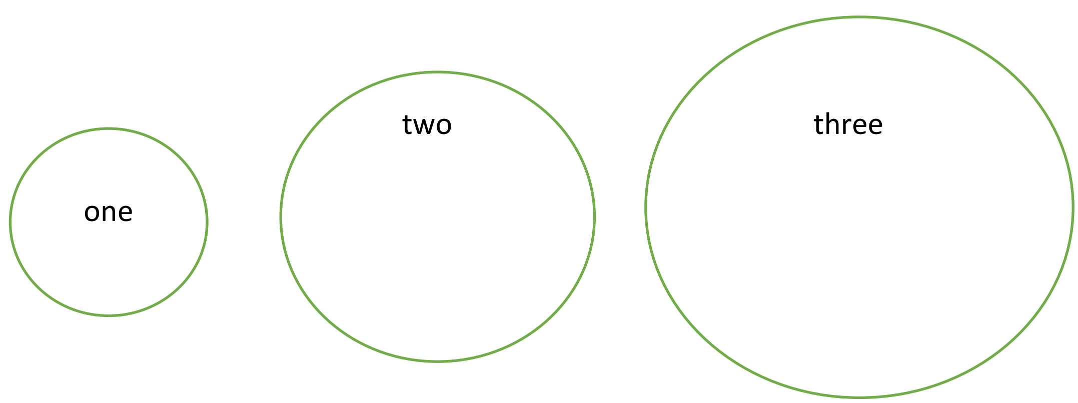
Comparing two sides of the **same triangle** (stay with one triangle) we see:
8/**16** = **1** / **2** which tells us that our smaller triangle's **red** side is half its **green** side!
Switching to our bigger triangle we see : **24**/**48 = 1** / **2.**

We can compare the corresponding sides **across our two** triangles like this.

Comparing corresponding sides of the **TWO triangles** we see:
8/24 = **16/48** = **1/3** which tells us that our bigger triangle is three time bigger than our smaller one! Bonus: We also know that **x/ y** = **1/3** without knowing the actual values for **x** and **y**!

Now, we look at circles! We zero in on three different sized circles and we consider three linear attributes of a circle. Things we can measure with our *flexible tape measur*e include the radius, diameter, and circumference.

1. The radius
2. The diameter
3. The circumference



Circle one	Diameter = D_1	Circumference = C_1
Circle two	Diameter = D_2	Circumference = C_2
Circle three	Diameter = D_3	Circumference = C_3

Using ratios to compare numbers we can compare the corresponding attributes **across our three** circle like this. Here, we compare diameters and circumferences.

$C_1/D_1 = C_2/D_2 = C_3/D$ ← this tells us that a circles' circumference divided by its diameter (a ratio) will always be the **same value** (in math terms the ratio C/D is a *constant*)

For a (any) circle : $C/D = X$ where X is a constant and $C = X * D$

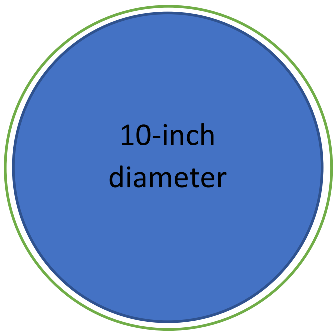
Of course, we do NOT use the variable name X for this constant! That would be a Math mortal sin! This **constant** is quite famous; a number that has fascinated mathematicians for centuries. The value even has its own name – the single Greek letter Pi or π. So, the circumference of any circle is $C = \pi * D$

For our readers who are engineers:

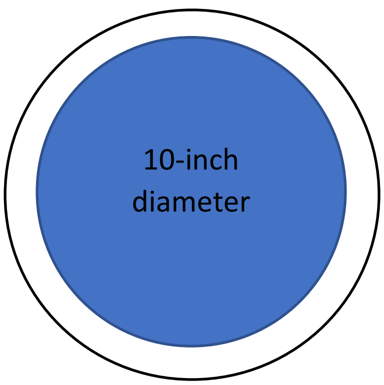
The approximate value for π is 3.14159265358979323846264338327950

Circumference example

Given a blue circle with diameter 10 inches. We wrap or surround the circle with a GREEN string. Excuse my diagram where we have not wrapped tight enough! We can straighten out the string and measure it with a ruler. The length of our string is the circumference of our blue circle or $10\pi \approx 31.4159265358979$ inches. We use ≈ to indicate 'approximately equal.'



Now we get another string - here a **BLACK string**. This time we wrap or surround our blue circle with 1 inch of space all around like this. Question, how much longer is our **BLACK string** compared to the GREEN string? Note: the **BLACK string** forms a circle of diameter 10+2=12



BLACK string length (circumference) = $(10 + 2)\pi = 10\pi + 2\pi$. Note: 10π is the length of the GREEN string. So, the **BLACK string** is 2π inches longer that GREEN string where 2π inches ≈ 6.283185307179586 inches.

Now suppose our BLUE circle is much bigger than 10 inches. Say it has the diameter of my favorite planet Saturn – some folks say I am from there! The last time I checked Saturn's **diameter** was 74,898 miles or 74,898 * 5,280 feet = 395,461,440 feet = ***4,745,537,280*** inches. After all that math, we see that Saturn's diameter is 474,553,728 times bigger that our original BLUE circle

To wrap Saturn, we need a GREEN String of length ***4,745,537,280*** π ≈ 14,908,545,056 inches!

Now, we wrap Saturn in a **Black string** with our 1-inch space.

BLACK string = (***4,745,537,280*** +2) π = ***4,745,537,280*** π + 2 π where ***4,745,537,280* π** is the length of the GREEN string.

So, the **BLACK string** is 2π inches longer that GREEN string where 2π inches ≈ 6.283185307179586 inches. ← Am I repeating myself here or is this déjà vu?

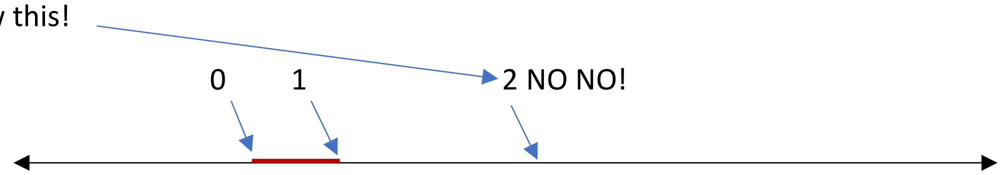
Compare this result with the 10-inch diameter example above. Does this seem 'intuitive' to you!?

The Number Line

We 'build' a number line.

We start with a line – like a geometry example. We then pick ANY point on the line and assign it the integer number 0 (zero); this point is called (named) the '*origin*'. We now pick 'any' point on the line to the **right** of zero. Here, we pick a point 'close to" zero so we can fit the diagram on one page of paper or one computer screen! We assign this point the integer number 1

We now have a 'scale.' Points 0 and 1 form a line segment on our line; the length of this segment is our ***unit*** for all subsequent scaling. The distance between 0 and 1 (a ***unit***) can be inches, miles, light years, kilometers - you decide!. Now it is a simple job to assign numbers (after all this is a number line!) to the number line. We add some points; first some integers. For two consecutive integers N and N+1 the distance between them must be one unit. We DO **NOT** allow this!

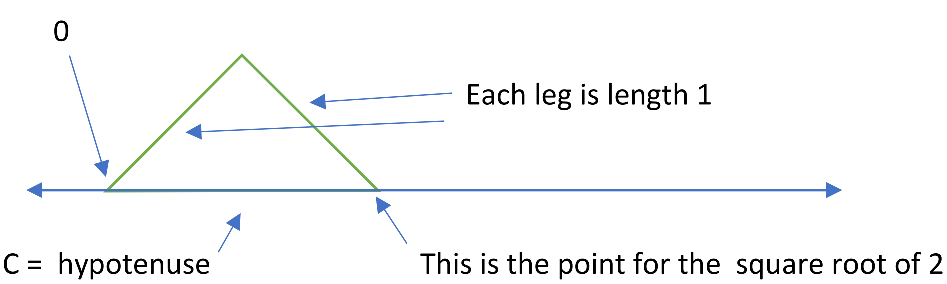


- Integer 2 must be one ***unit*** to the right of 1.
- Integer 3 is one ***unit*** to the right of 2
- Etcetera
- Integer -1 is one ***unit*** to the left of 0.

Number 3½ is halfway between 3 and 4.

The square root of 2 is a number.

We can find the ***point*** on a number line for square root of 2. We construct a 45-45-90 right triangle whose legs are one ***unit***. We place it with its hypotenuse on our number line and with the left-most acute angle vertex point on point zero

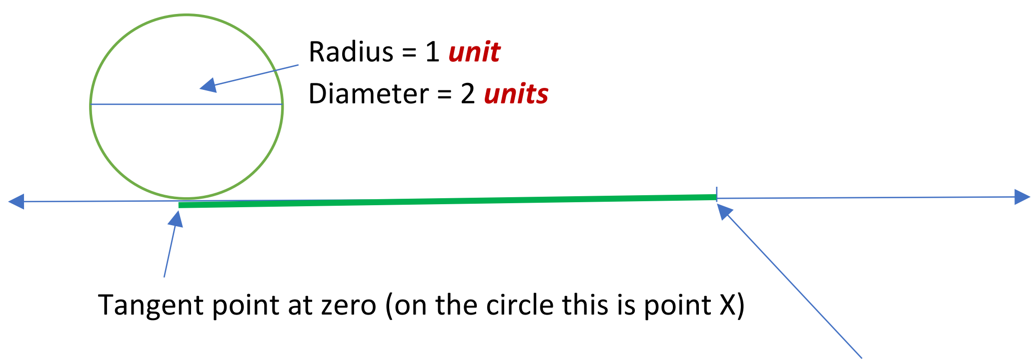


We are done! Why? Well! Let us calculate the length of the hypotenuse via Pythagorean theorem $A^2 + B^2 = C^2$

$C^2 = 1^2 + 1^2$
$C^2 = 1 + 1$
$C^2 = 2$
C = square root of 2

We can also find the point for the number 2π; the circumference of the unit circle - the circle with radius 1.

Construct a unit circle that is tangent to the number line at point zero. Label the point of intersection on the circle as X. The circumference of our circle is 2π. Now, think of the circle as a wheel and perform one full rotation to the right (clockwise). The circle's point X will end up back on the number line at distance 2π from the point zero.



X is the point for 2π

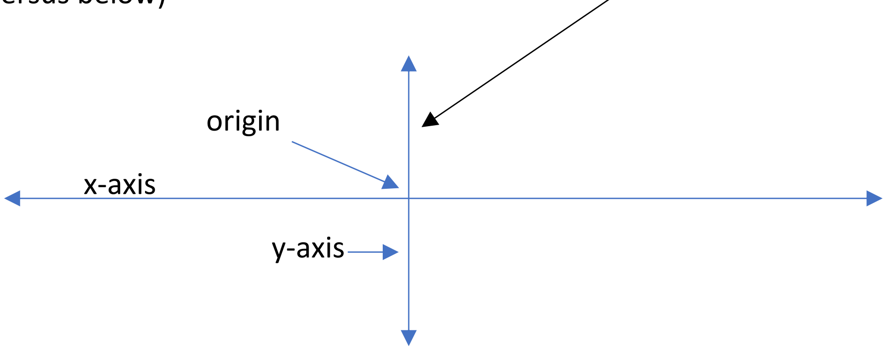
If you are looking for π, perform ½ circle rotation or bisect the green segment!

The Cartesian plane

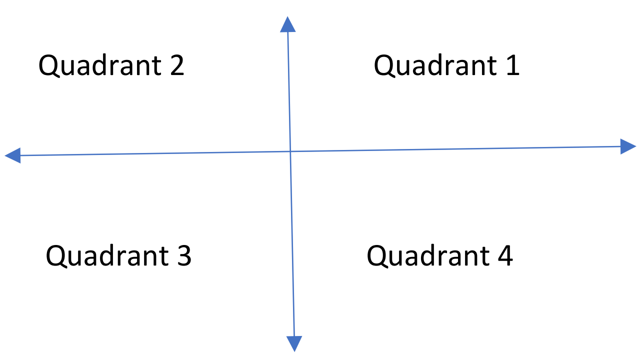
René Descartes (his name in Latin is *Cartesius*) was a 17th century Math enthusiast who 'invented' what we now call the *cartesian coordinates system* or the *cartesian plane*. Like many amazing and brilliant ideas, his was a simple one. In the world of Math simple if often a prerequisite for brilliance! A simple solution to a problem is a brilliant solution. The invention of a simple but useful tool is brilliant – e.g., the wheel or the cartesian plane. Repeat after me; in engineering the rule is ALWAYS : ***the simplest solution is the best solution***!

René took a plane from geometry and constructed two number lines (each line is called an axis) per the following diagram.

- The axes are perpendicular (form a right angle)
- The axes intersect each other at their respective *origins.* The intersection point is called the plane's *origin*
- The ***unit*** on the x-axis MUST be the same length as the ***unit*** on the y-axis
- The vertical line (axis) is oriented such that its *positive numbers* are above the x-axis (versus below)



By convention, the horizontal number line is called the x-axis and the vertical line the y-axis. The plane is cut into four *quadrants*. Points in the plane can be located by the traditional (X,Y) coordinates we are all familiar with.



A point in the Cartesian plane can be located via its XY-coordinates. We need two numbers to locate one point; the two numbers are called the point's *coordinates*. By convention, we ALWAYS list the X-axis point first **and** place the two numbers inside a pair of round brackets **and** separate by a comma like (X,Y)

To locate a point (X,Y) . You can switch the order of step two and step three!

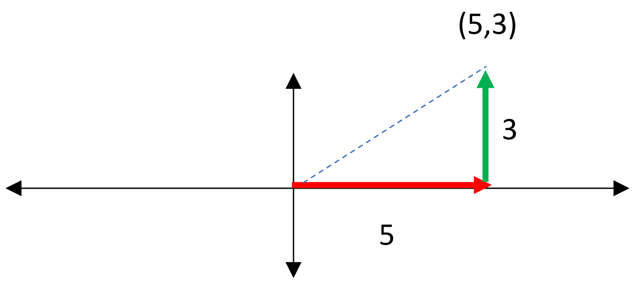
1. Step one: Start at the origin – point (0,0)
2. Step two: Move horizontal

   Move (right or left) a distance X.
   The distance may be ZERO.
   Move right if $X > 0$ OR left if $X < 0$

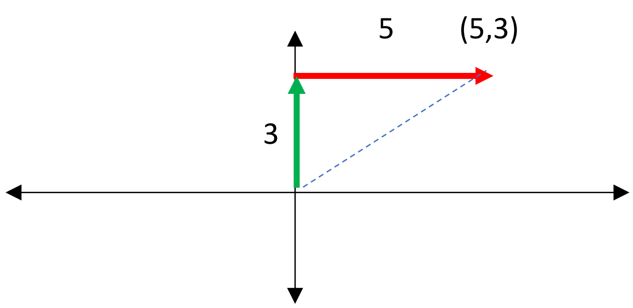
3. Step three: Move vertical

   Move (up or down) a distance Y
   The distance may be ZERO.
   Move up if $Y > 0$ OR down for $Y < 0$

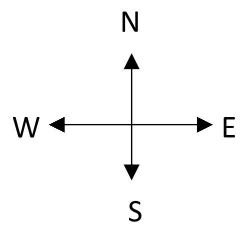
Notice that we move horizontally and/or vertically - so (X,Y) are also called *rectangular coordinates*. Example, to locate point (5,3). We move RED than GREEN.



You can switch the order of step two and step three! We move GREEN than RED



Maps often have this neat little aid.



You can think of our X-Y Plane as a huge map. To locate a point given its coordinates, we always start at the origin (0,0). Earlier we 'moved' horizontally right or left and then vertically up or down.

- Moving right or left is the equivalent to east or west
- Moving up or down is the equivalent to north or south

To locate a point (X,Y) . You can switch the order of step two and step three!

1. Step one: Start at the origin – point (0,0)
2. Step two: Move horizontal

   Move (east or west) a distance X.
   The distance may be ZERO.
   Move east if X > 0 OR west if X < 0

3. Step three: Move vertical

   Move (north or south)a distance Y
   The distance may be ZERO.
   Move north if Y > 0 OR south for Y < 0

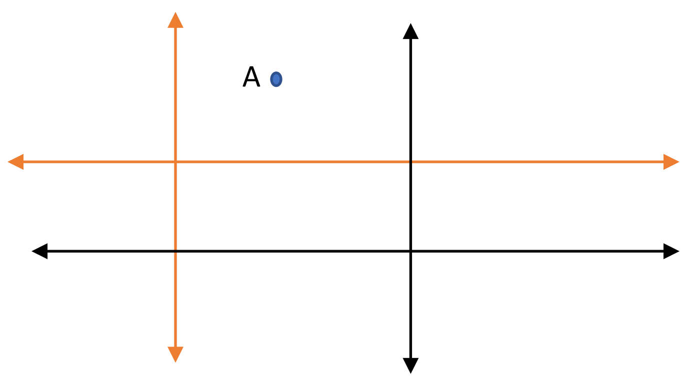
Start of the origin (0,0)

Coordinates	***Horizontal: east or west***	***Vertical: north or south***	***Location***
	Step 1	***Step 2***	
(3,5)	Move east 3 ***units***	Move north 5 ***units***	Quadrant 1
(-4, 19.5)	Move west 4 ***units***	Move north 19.5 ***units***	Quadrant 2
(-10, -2)	Move west 10 ***units***	Move south 2 ***units***	Quadrant 3
(8,-8)	Move east 8 ***units***	Move south 8 ***units***	Quadrant 4
(6,0)	Move east 6 ***units***	Do not move	On the positive x-axis
(-6,0)	Move west 6 ***units***	Do not move	On the negative x-axis
(0,9)	Do NOT move	Move north 9 ***units***	On the positive y-axis
(0,-9)	Do NOT move	Move south 9 ***units***	On the negative y-axis

Moving things around.

Note, a plane **cannot** move; it is where it is and that is the end of the story!. A point on the plane **cannot** move; a point is a location on a plane that **cannot** move!

Example: the paper or computer screen you are currently reading represents a plane; point A is on the plane. We can draw our X-axis Y-axis lines anywhere on this plane. When we do we can label point A with a pair of (X,Y) coordinates. In the following diagram I could not make up my mind and ended up constructing two {x-axis, y-axis} pairs – **orange** and **black**. Clearly A has not (cannot) move in our original plane. Point A has two **different** (X,Y) coordinates depending on which {X-axis, Y-axis} pair we end up using. Like I say to my nieces and nephews: “it is all relative”

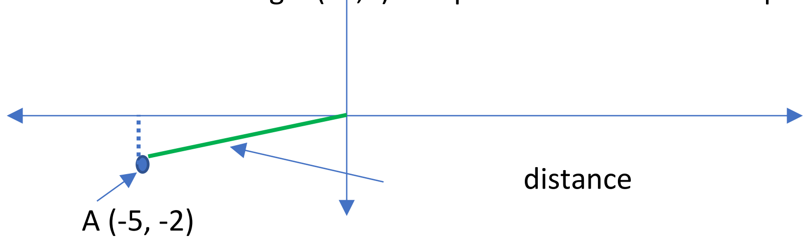


Distance in the Cartesian Plane

For two points in the Cartesian plane, we can measure the distance between them.

Example:

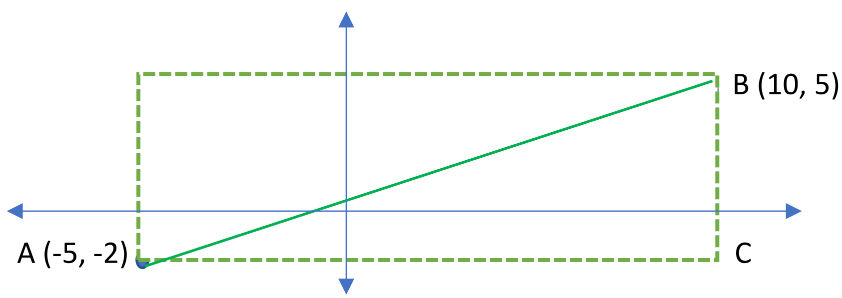
calculate the distance between the origin (0 ,0) and point A in the cartesian plan



Consider the right triangle above with vertex points (-5,2) (0,0) and (-5,0). The two legs of our triangle have length 2 and 5. Apply the Pythagorean theorem

$distance^2 = 5^2 + 2^2$
$distance^2 = 29$
$distance = \sqrt{29}$

Another example



Construct the above rectangle. We keep everything ‘rectangular,’ i.e., the rectangle sides are all parallel and/or perpendicular to one of the axes. The distance AB is the hypotenuse of two right triangles. Examining A and B coordinates we see that Point C has coordinates (10, -2) .

Length AC = 5+10 = 15. Length BC = 5+2 =7

Apply the Pythagorean theorem

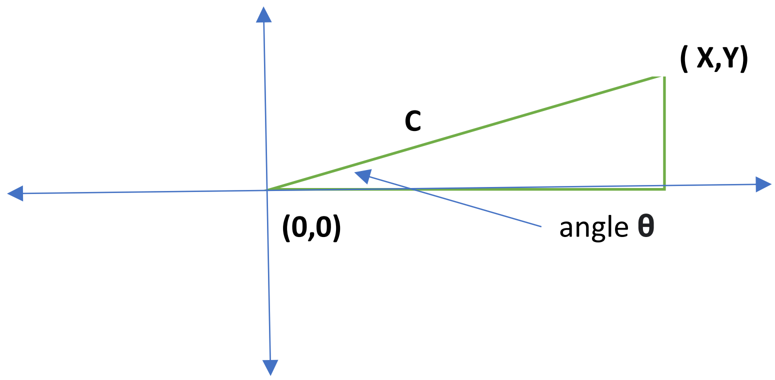
$distance^2 = 15^2 + 7^2$ note: the distance is also the rectangles diagonal
$distance^2 = 225+49$
$distance^2 = 274$
$distance = \sqrt{274}$
distance = 16.55294535724684859 note $\sqrt{274}$ is in between $16^2 = 256$ and $17^2 = 289$

This generic formula is valid:

For point (x_1, y_1) and (x_2, y_2) the distance is: square root ($(x_2 – x_1)^2 + (y_2 – y_1)^2$)

Geometric shapes in the Cartesian Plane

René’s discovery allows us to **position** a right triangle or any other geometrical shape in the cartesian plane – as opposed to the ‘plane’ used by Euclid and other ancient mathematicians. By convention, we often **position** a right triangle like this.



We **name** the angle at the origin with the Greek letter **θ (**theta). Recall the Pythagorean theorem $\mathbf{X}^2 + \mathbf{Y}^2 = \mathbf{C}^2$

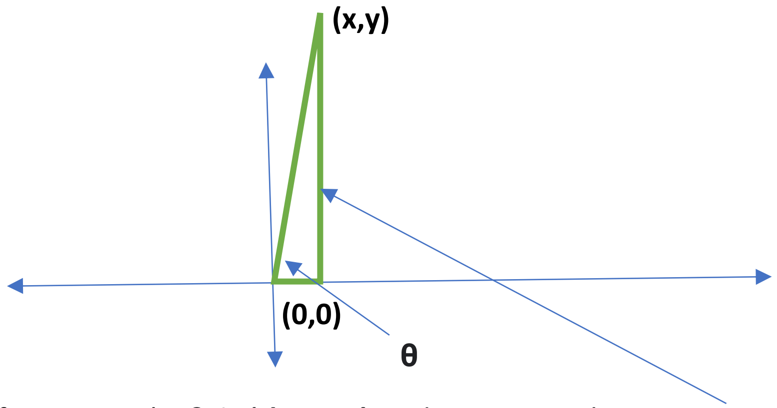
Using **θ** (at the ‘origin’) as a reference, we call (**name**)

- the side with length **Y** the ‘***opposite’*** side (opposite or across from **θ)**
- the side with length **X** the ‘***adjacent’*** side (adjacent or next to **θ)**
- the side with length **C** (opposite the right angle) is called the ***hypotenuse***

Earlier, we discussed comparing the three sides of a single triangle via ratios. Specifically for right triangles with sides a,b and c, we can evaluate six ratios: a/b a/c b/a b/c c/a c/b. For now, we zero in on three of the six – here they are again:

sine(**θ**) = opposite / hypotenuse or **Y/C**
cosine(**θ**) = adjacent / hypotenuse or **X/C**
tangent(**θ)** = opposite / adjacent or **Y/X**

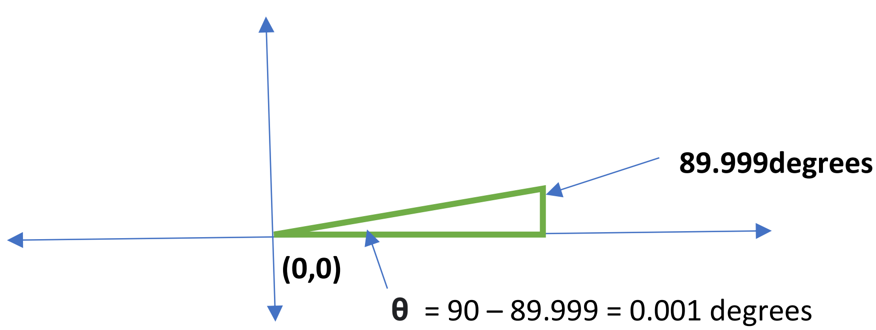
Example



Here our reference angle **θ is 'close to'** 90 degrees. So, the opposite side is 'almost' the same size as our hypotenuse. With my calculator I evaluated the following sine (opposite/hypotenuse) values for these angles (4 different values for angle **θ**).

Angle* θ *measures	***Approximate value of sine*(θ)**
85 degrees	0.99619469809174553229501040247 39
89 degrees	0.9998476951563912391570115588139
89.999 degrees	0.9999999998476912901105120241781
89.999999 degrees	0.9999999999999998476912901066457

Now, we take our triangle and reposition it as follows



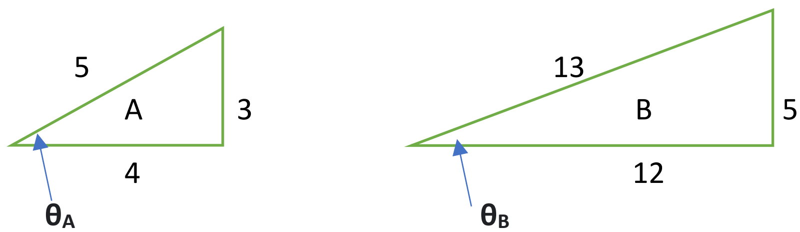
With my calculator I evaluated the following

cosine (0.001) = 0.99999999984769129011051202417815

It should NOT surprise you that the cosine(90 - 89.999) = sine (89.999). If it does, think about it!

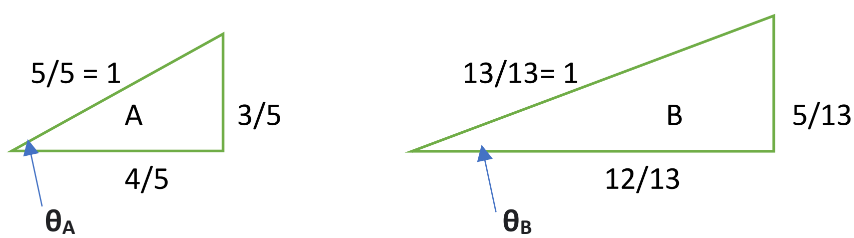
Shrinking the hypotenuse

The following two **right** triangles are examples of 'Pythagorean triplets;' their sides are all ***integer*** values which makes them easier to work with. {3,4,5} and {5,12,13} are two examples of these triplets.



In triangle A sine(θ_A) = 3/5
In triangle B sine(θ_B) = 5/13

We can shrink our triangles - they will no longer be Pythagorean triplets. Here we shrink just enough to make our two hypotenuses have length 1. Note the angles θ_A and θ_B do **NOT** shrink!



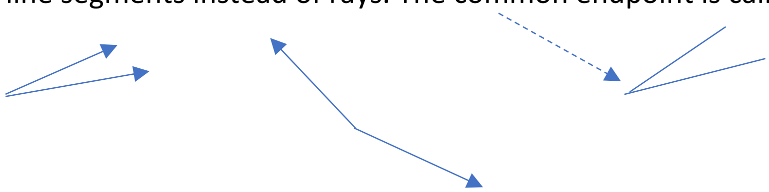
Of course, there is NO change to the following 2 ratios ! Our hypotenuse is 1 so dividing has 'NO effect' – in general K/1 = K for any number K!

In triangle A sine(θ_A) = $\text{opposite}_A/\text{hypotenuse}_A$ = 3/5
In triangle B sine(θ_B) = $\text{opposite}_B/\text{hypotenuse}_B$ = 5/13

Measuring angles

Here is a ray with endpoint A

An angle is the intersection of two rays with the same endpoint. We can take a shortcut and use line segments instead of rays. The common endpoint is called the angle's *vertex.*



We all know how to measure things – we use a ruler for linear things and a protractor for angles.

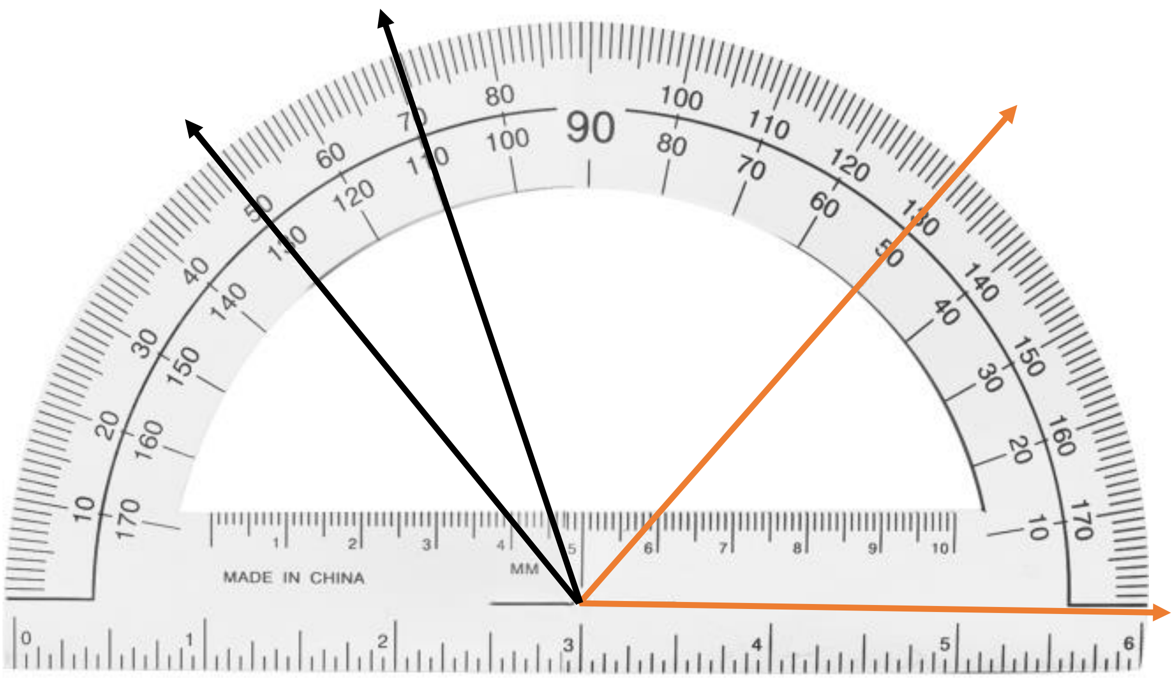
In general, when we measure two objects (a line segment, a circle, a backyard, a basketball player height) the measurement must be 'meaningful.' When we measure we assign a number as the 'measurement.' For example, our point guard's height is 75 inches. Given the following diagram, ff we measured players heights we expect our point guard's height to be less than our center's height – if not our measurement method is suspect at best.

Point Guard **Center**

Using The Protractor

We are (we better be) all familiar with a protractor. We will do a quick review.

Here is a picture and example. We measure two example angles and discover that they measure 50 degrees (the **ORANGE** angle) and 20 degrees (the **BLACK** angle)



We take a circle (the protractor maker cut it in half to save plastic!). On the circle we mark 360 points equally distant from each other – making 360 arcs all equal in size. Each arc is 1/360th of the circle circumference. I call these miniARCS. Because all circles are similar, the circle can be of any size. We measure an angle by determining how many (the count of) mini-ARCs the angle 'takes up.'

We do NOT consider the length of a mini-ARC – that length would depend on the size of the protractor (of the circle). Measure an angle with two different protractors of varied sizes and you (MUST!) get the same measurement.

If you are a curious person or a cat you may want to know where the 360 number came from!

A bit of history. The ancient and very clever Sumerians in the 3rd millennium BC used a sexagesimal number system. This is a base 60 system (it has 60 unique symbols or combinations of symbols for its 60 digits!). Before you even ask; the answer is NO – they did NOT have 60 fingers.

We write $360_{10} = 3 * 10^2 + 6 * 10^1 + 0 * 10^0$
They write $60_{60} = 6 * 60^1 + 0 * 60^0$

360_{10} and 60_{60} represent the same (they are equal) number!

The equally clever Babylonians inherited this system **and** invented the protractor.

There is something else about 360 that we like! It is 'highly composite' – it can be broken down (factored) into A*B where A and B are integers – using 'many' different A, B pairs.

Here is a mini-Python program. We are looking for integer factors A and B where A*B = 360. We found 12 of them! Hers is the source code.

```
count = 1;
a = 2;
MAXa = 180;

while (a<MAXa):

    if (360 % a == 0):

        b = int(360/a);
        print (count , ') ',  a, ' * ' , b , ' = 360');
        count = count + 1;
        MAXa=b;

    a = a + 1;
```

Output:

1) 1 * 360 = 360
2) 2 * 180 = 360
3) 3 * 120 = 360
4) 4 * 90 = 360
5) 5 * 72 = 360
6) 6 * 60 = 360

7) 8 * 45 = 360
8) 9 * 40 = 360
9) 10 * 36 = 360
10) 12 * 30 = 360
11) 15 * 24 = 360
12) 18 * 20 = 360

Thanks again to the Sumerians and Babylonians.

In geometry we typically see angles in the *range* [0, 360). If we combine Geometry with some Algebra related problems we may end up calculating an angle measurement outside of this *range*. For example, if we calculate an angle of 365 degrees we can 'massage' the measurement by subtracting 360; here are some examples. Note we either add OR subtract 360 !

Angle (degrees)	Massage	Massaged Angle (degrees)
367	367 - 360	7
788	788-360 = 428 428 - 360 = 68	68
-35	-35 + 360 =325	325
-899	-899 + 360 =-539 -539 + 360 = -179 -179 + 360 = 181	181
-1	-1 + 360 = 359	359
-89	-89 + 360 = 271	271
-90	-90 + 360 = 270	270
-91	-91 + 360 = 270	269
360	360 - 360 = 0	0
361	361 - 360 = 1	1

A more elegant solution via a mini-Python program and the Python remainder operator %

```
angleList = [367, 788, -35, -899, -1, -89, -90, -91, 360, 361];

for x in angleList:

    y = x%360

    print (str(x), str(y));
```

Output:

```
367   7
788   68
-35   325
-899 181
-1   359
-89   271
-90   270
-91   269
360   0
361   1
```

Using Unit Circle ARCs

We now look at a second method for measuring angles.

To measure an angle, we must assign a meaningful number that represents the size of the angle.

For example, clearly angle A must measure bigger than angle B.

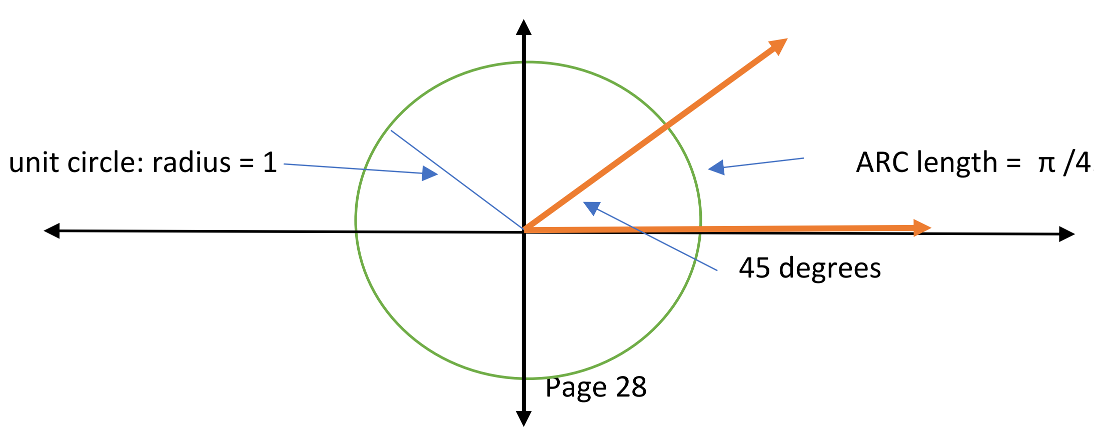
Our second method is like the protractor method; it also uses use a circle – BUT not any run of the mill circle. This method uses the 'unit circle' ; the circle whose radius is ONE. Earlier, we discovered the formula for the circumference of a circle C = π * D. For the unit circle D = 2 so the formula becomes C = 2 π

- **Remember this : the formula for the circumference of a circle is D*π**
- **Remember this: the circumference of the unit circle is 2 π**

Here is a picture with our unit circle in the Cartesian plane conveniently (for us) centered at the origin. We place an **(orange) angle** with its vertex at the origin and one ray along the positive x-axis. We note that the angle intersects the circle at two points – one point if the angle is ZERO degrees. The **angle** is in the plane's first quadrant. The two intersecting points mark off an ARC on the circle - a subset of its circumference. The circle's circumference is = 2π.

To measure the **angle**, we must assign a meaningful number that represents its size So, we use **the length of the ARC** as the **angle** measure. Do not allow π to scare you; it is just a number! To get a bit more concrete, our unit circle's circumference = 2π is approximately 2* **3.14159** = 6.28318.

Here is a 45-degree **angle** and a unit circle. Since 45/360 = 1/8, our ARC length is 1/8th of the unit circle's circumference. ARC length = 1/8 * 2π = π/4.



Note: π /4 is 'just' a number like 45 is a number…do not let the symbol π scare you! Here are some ARC lengths

Degrees	Notes	ARC length
1	1 degree is 1/360th of the circumference, so 2 π /360 = π /180	π /180
180/π	180/π * 1/360 = 1/2π therefore 180/π degrees intersects (1/2π)th of the circumference where 1/2π * 2π = 1	1
45	45 * 1/ 360 = 1/8 therefore 45 degrees intersects 1/8th of the circumference where 1/8 * 2 π = π/4	π /4
90	90 degrees intersects 1/4th of the circumference, so 2π * 1/4 = π/2	π /2
180	180 degrees intersects 1/2 of the circumference, so 2π * 1/2 = π	π
270	270 degrees intersects to 3/4th of the circumference, so 2π * ¾ = 3π /2	3 π /2
360	360 degrees corresponds to the circumference	2 π

Note :

- π **/180** ARC length *corresponds to* 1 degree
- 1 ARC length *corresponds to* 180/π degrees (approximately 57.2958279 degrees)

Since **π /180** ARC length corresponds to 1 degree, we can use this relationship to convert degrees to a unit circle ARC length.

Degrees to ARC length conversions

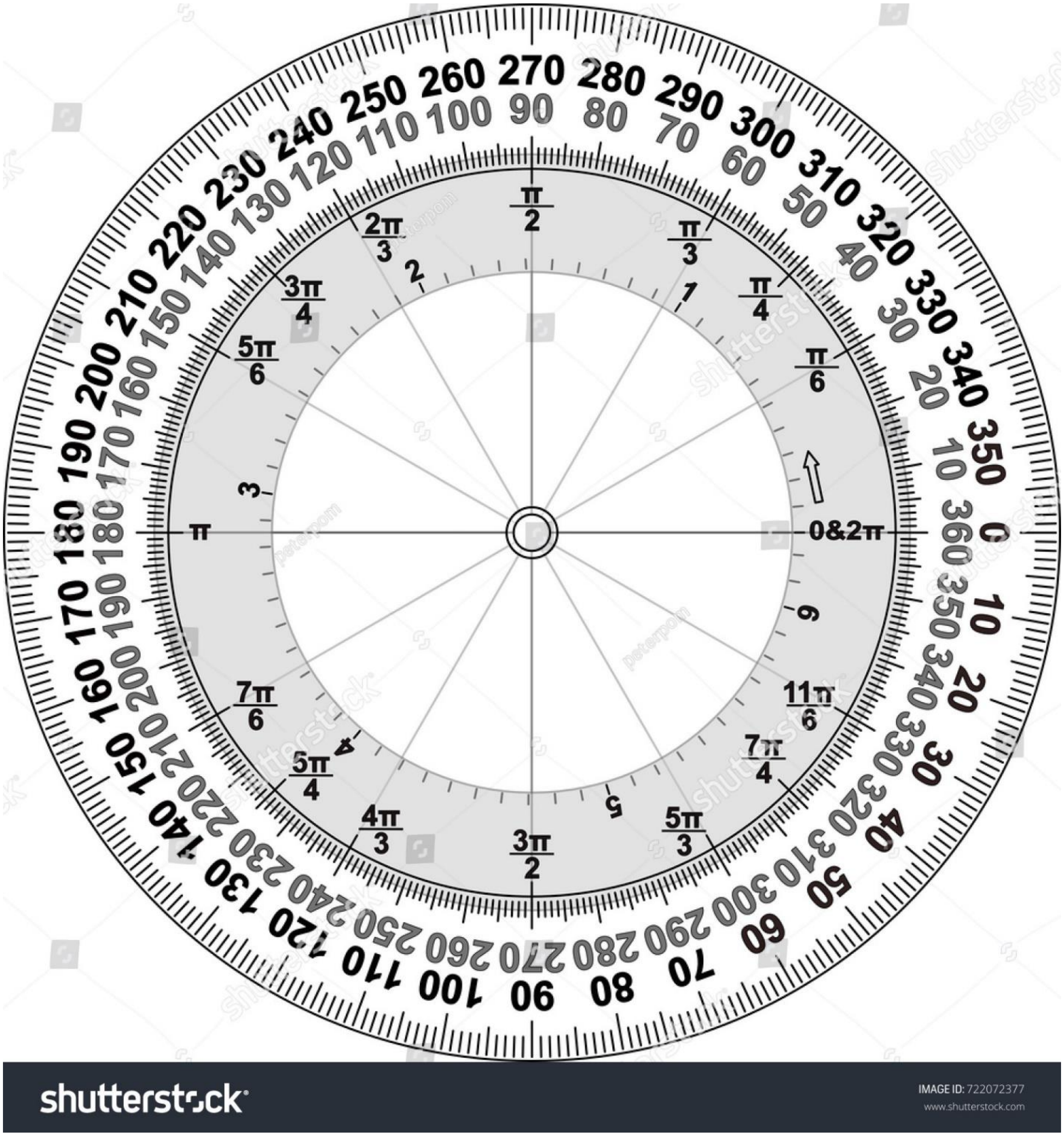
Degrees	Conversion math	ARC length on unit circle
0	0 * π /180	0
1	1 * π /180	π /180
30	30 * π /180	π /6
45	45 * π /180	π /4
90	90 * π /180	π /2
180	180 * π /180	π
270	270 * π /180	3π /2
360	360 * π /180	2π

Since ARC length of 1 *corresponds to* 180/π degrees we can use this relationship to convert ARC lengths to degrees. Simply multiple the ARC length by 180/π. For example, the ARC length of π (half of the unit circle) is converted via: π * 180/π = 180 degrees.

ARC length to degrees conversions

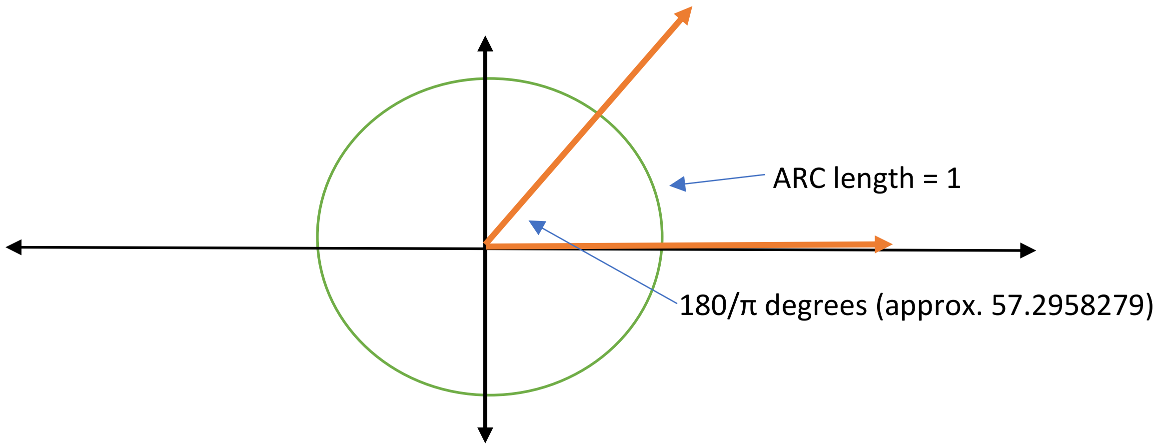
ARC length	Conversion to degrees	Degrees
0	0 * 180/π = 0	0
π/6	π/6 * 180/π = 180/6 = 30	30
π/4	π/4 * 180/π = 180/4 = 45	45
π/3	π/3 * 180/π = 180/3 = 60	60
1/2 π	1/2 π * 180/π = 1/2*180 = 90	90
3/4 π	3/4 π * 180/π = 3/4*180 = 135	135
5/6 π	5/6 π * 180/π = 5/6*180 = 150	150
π	π * 180/π = 180	180
7/6 π	7/6 π * 180/π = 7/6*180 = 210	210
5/4 π	5/4 π * 180/π = 5/4*180 = 225	225
4/3 π	4/3 π * 180/π = 4/3*180 = 240	240
3/2 π	3/2π * 180/π = 3/2*180 = 270	270
5/3 π	5/3 π * 180/π = 5/3*180 = 300	300
7/4 π	7/4 π * 180/π = 7/4*180 = 315	315
11/6 π	11/6 π * 180/π = 11/6 *180 = 330	330
2 π	2π * 180/π = 360	360

Following is an image of a protractor that measures both degrees and radians. For example, when we say an angle measures π /4 radians we are referring to an angle whose unit circle ARC is $1/8^{th}$ of the circumference. 45 = 1/8 * 360 degrees.



Thanks shutter stock!

Example: We can measure an ARC length of length 1 with our *flexible tape measure*. We multiple the ARC length by 180/π to convert to degrees



The Radian

To measure angles, the Babylonians divided up a circle into 360 ARCs (**degrees**) of equal length.

I will refer to one such ARC as a 'mini-ARC.' A mini-ARC corresponds to one degree.

So, when we say an angle measures '45 degrees' we are referring to an angle that intersects an ARC that 'takes up' 45 mini-ARCs . Note: 45 degrees is 1/8th of the circle.

Alternately we can use the ***unit circle*** intersected ARC to measure angles.

We simply measure the length the ARC with our *flexible tape measure.*

We now have 2 methods for measuring angles. We are now vague if we say an angle is 2. Does the 2 refer to 2 mini-ARCS (degrees) or does it refer to an ARC of length 2 on our unit circle? So, to be precise we say the angle is **2 degrees** or we say the angle is **2 radians**. Radians is a new term that refers to the angle measurement mechanism that uses the ***unit circle.***

Since the unit's circle circumference is 2π we often see angle measurements like π /4 radians or π radians or π /8 radians

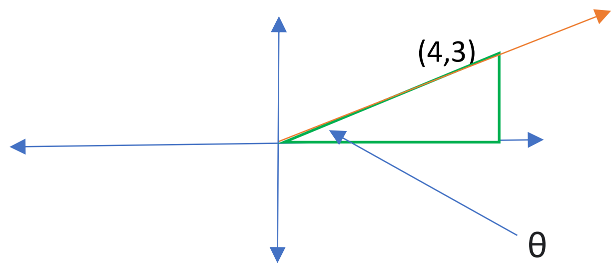
The Trigonometric ratios

Earlier we compared the sizes of a single right triangle using ratios. We gave names (sine, cosine, and tangent) to three of the ratios.

We observed that a triangle's size (but NOT shape) can change (we can enlarge or shrink a triangle) without changing the ratios of its sides – similar triangles.

We also, discussed the convention for placing a right triangle in the Cartesian plane.

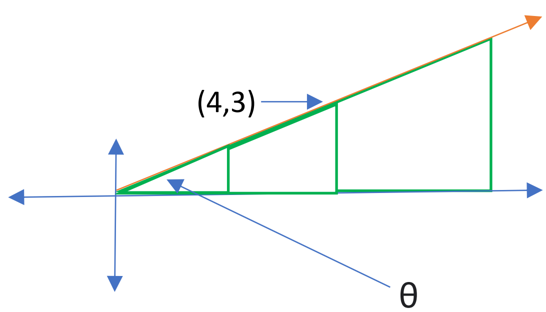
Example of a {3,4,5} right triangle in the plane and the conventional position in the cartesian plane



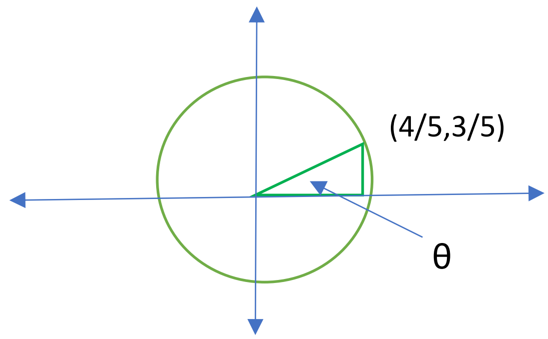
Remember {3,4,5} is a Pythagorean triplet

- sine(θ) = opposite/hypotenuse = 3/5
- cosine(θ) = adjacent/hypotenuse = 4/5
- tangent(θ) = opposite/ adjacent = 3/4

Shrinking and/or enlarging our triangle has NO effect on the above ratios. The following three right triangles have the same sine, cosine, and tangent ratios - the same shape.



Consider our smallest triangle. We shrank the origin {3,4,5} triangle! If we shrink such that the hypotenuse is 1 we produce the following triangle with hypotenuse 1 - which means that the point (4/5, 3/5) is on the unit circle which we include in the following diagram!



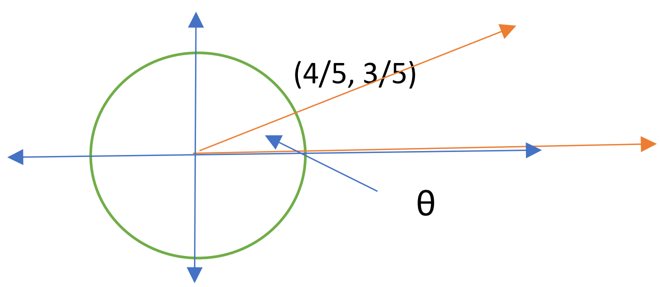
We now remove the triangle and focus on the point (4/5,3/5) and the angle θ.

By convention we place an angle (here θ) in cartesian plane in the following position – no surprise! The point (4/5,3/5) is the point where our angle intersects the unit circle. Notice the coordinates of the points of the unit circle; for any such point (X,Y)

0 <= X <=1 and 0 <= Y <=1

The four points where the circle intersects the X and Y axes are:

(1,0) (0,1) (-1,0) (0,-1)

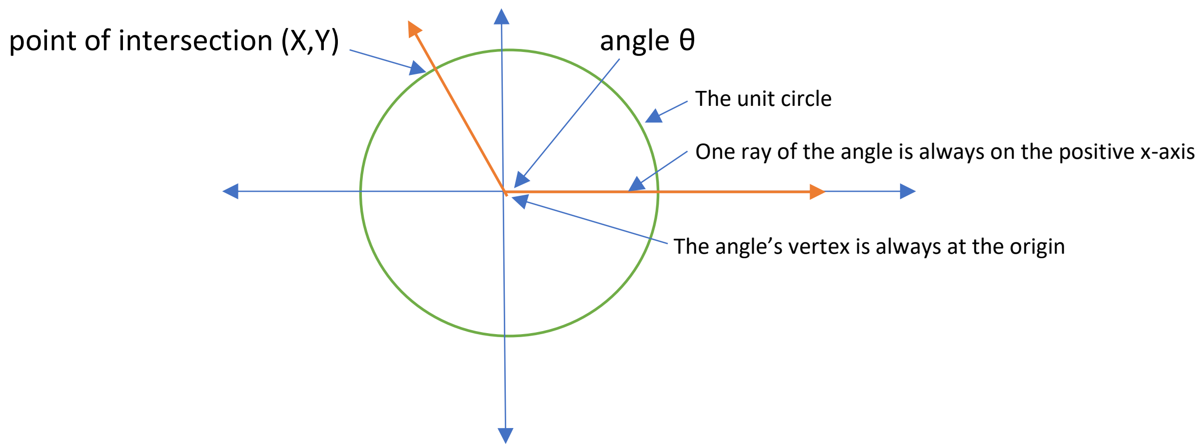


Our ratios survived all my babbling!

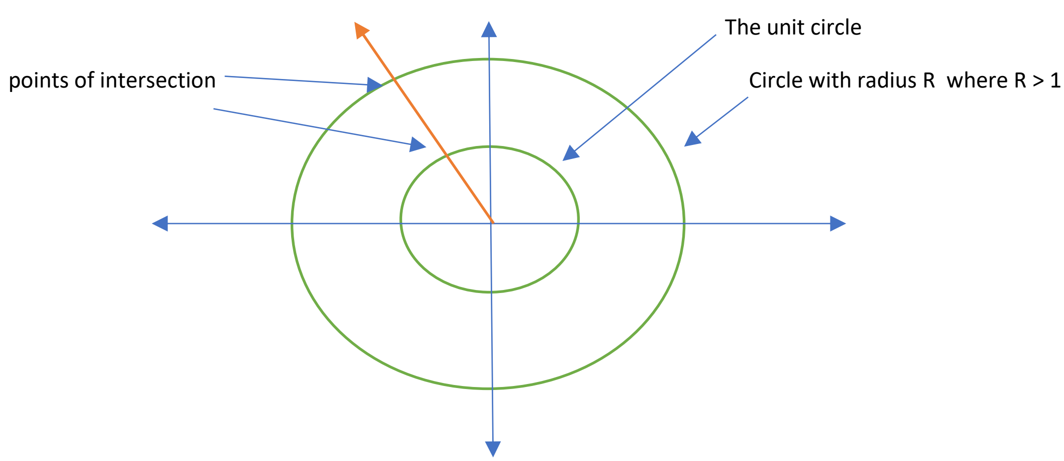
- sine(θ) = y coordinate of the intersecting point= 3/5
- cosine(θ) = x coordinate of the intersecting point= = 4/5
- tangent(θ) = y coordinate / x coordinate = (3/5) / (4/5) = (3/5)*(5/4) = 3/4

By removing the triangle, we can generalize and 'allow' *our point of intersection* to be any point on the unit circle – here we call it point (X,Y)). The point can be **anywhere** on the unit circle. We define our ratios.

- sine(θ) = Y
- cosine(θ) = X
- tangent(θ) = Y/X for X not zero!



We are NOT limited to a unit circle - here is a larger circle - its radius is greater than 1

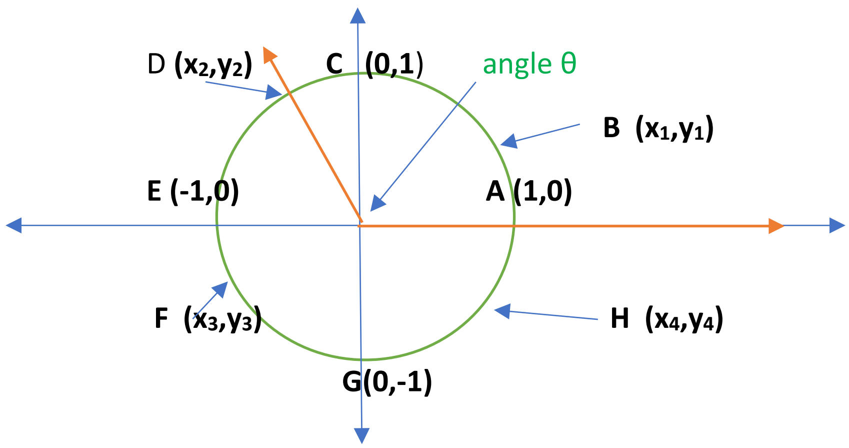


If our points of intersection are (X_1, Y_1) for the unit circle and (X_2, Y_2) for our larger circle we have

- sine(θ) = $Y_1/1$ OR sine(θ) = Y_2/R
- cosine(θ) = $X_1/1$ OR cosine(θ) = X_2/R
- tangent(θ) = Y_1/X_1 OR tangent(θ) = Y_2/X_2

Envision the angle θ (our orange angle below) starting at 0 degrees and moving counterclockwise around the unit circle. One ray of our angle moves, and one ray remains stationary positioned along the x-axis. As we rotate, we keep the vertex point at (0,0). Our diagram shows the angle intersecting our point D in the second quadrant.

Counterclockwise



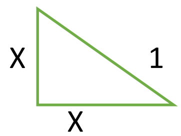
As we rotate, our angle intersects our labeled points, and the ratios change.

Point	*Coordinates*	**θ in degrees**	*sine(θ)*	*cosine(θ)*	*tangent(θ)*
A	(1,0)	0	0	1	0
B	(x_1,y_1)	$0 < \theta < 90$	y_1	x_1	y_1/x_1
C	(0,1)	90	1	0	undefined
D	(x_2,y_2)	$90 < \theta < 180$	y_2	x_2	y_2/x_2
E	(-1,0)	180	0	-1	0
F	(x_3,y_3)	$180 < \theta < 270$	y_3	x_3	y_3/x_3
G	(0,-1)	270	-1	0	undefined
H	(x_4,y_4)	$270 < \theta < 360$	y_4	x_4	y_4/x_4

Example

A 45-degree angle and its ‘friends’ (integer multiples) are easy to work with! We look at the 45-degree angle

Here is a 45-45-90 triangle with hypotenuse 1. It is an isosceles triangle



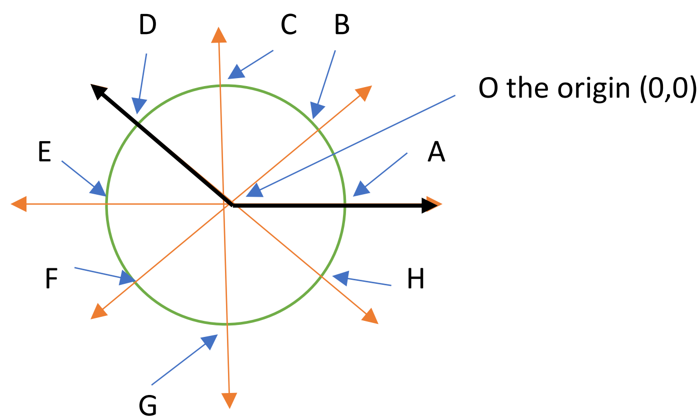
Solving the Pythagorean theorem

$X^2 + X^2 = 1^2$
$2X^2 = 1$
$X^2 = 1/2$
$X = 1 / \sqrt{2} = 1/ \sqrt{2} \ * \ \sqrt{2}/\sqrt{2}$
$X = \sqrt{2} / 2$
$X \approx 1.4142135623730950488016887242097 / 2$
$X \approx 0.70710678118654752440084436210485$

The friends of our 45-degree angle are the following eight angles. The **BLACK** ray is common for all 8 angles. I have marked the **135**-degree angle in BLACK.

{0, 45,90,**135**,180,225,270,315, …}

Notice that 135-degree ∠AOD intersects the unit circle at the point (-√2/2, √2/2) in the second quadrant.



Here are all eight angles and their sines and cosines. Notice Y coordinates in quadrants 1 and 2 are ALWAYS positive (Y > 0) and always negative in 3 and 4.

Angle	In degrees	In Radians	Y coordinate (sine)	X coordinate (cosine)	sine (green => approx.)
∠AOA	0	0	0	1	
∠AOB	45	π/4	√2 / 2	√2 / 2	0.70710678118654752440084436210485
∠AOC	90	π/2	1	0	
∠AOD	135	¾ π	√2 / 2	-√2 / 2	0.70710678118654752440084436210485
∠AOE	180	π	0	-1	
∠AOF	225	5/4 π	-√2 / 2	-√2 / 2	- 0.70710678118654752440084436210485
∠AOG	270	3/2 π	-1	0	
∠AOH	315	7/4 π	-√2 / 2	√2 / 2	- 0.70710678118654752440084436210485

The Trigonometric ratios (functions of our angle θ) cause nightmares in some students and agita for most high school educators. The issue – in part – is that the cumbersome "Trig identities' and formulas often require memorization!

So, to be kind we will discuss just one such identity. We discussed this earlier when we examined the Trigonometric ratios in a right triangle. Here it is again in our generalized version (without the right triangle) of the Trigonometric ratios

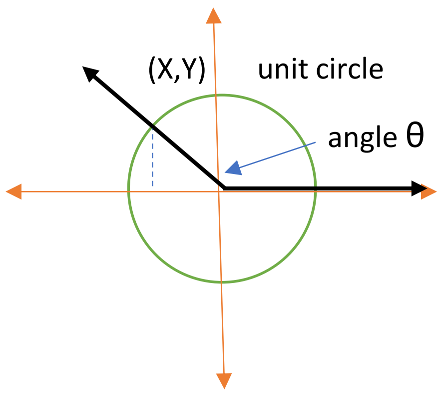
For any angle θ:

$$\text{sine}^2(\theta) + \text{cosine}^2(\theta) = 1$$

Re the exponent 2 above and to make things super extra clear, we can write

$$\text{sine}(\theta) * \text{sine}(\theta) + \text{cosine}(\theta) * \text{cosine}(\theta) = 1$$

This is NO BIG DEAL – it is simply a variation of the Pythagorean theorem! In the following diagram, we position our angle θ. We then construct a right triangle via our dotted vertical line and Pythagoras tells us that $X^2+Y^2 = 1^2 = 1$ where Y = sine(θ) and X = cosine(θ). Convince yourself that this all works in all four quadrants and for 0, 90, 180, 270 and 360 degree angles



Round and round we go where we stop nobody knows!

We first discussed the Trigonometry as the ratios of the three sides of a *right triangle*. We removed the triangle and defined the ratios via the XY-coordinates of the 'point of intersection' with the unit circle - like the above diagram. This allows us to determine the sine of θ where θ is in the second quadrant - we could not do that when we used the *right triangle* approach.

Now we take the concept one step further and expand the range of θ to $(-\infty , +\infty)$

Example

These are all equal - notice the pattern!

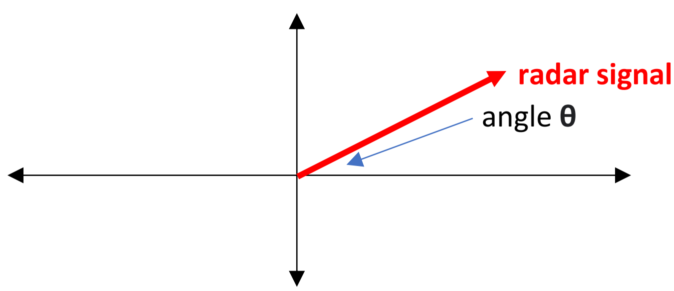
sine(π/4 - 2 π) = sine(π/4 - 4 π) = sine(π/4 - 2 π) = **sine(π/4)** = sine(π/4 + 2 π) = sine(π/4 + 4 π)

Polar Coordinates

Background

Its dark – nighttime. We can barely see. We lost our puppy! She is somewhere on the Cartesian plane, but we do not know where. We need to find her!

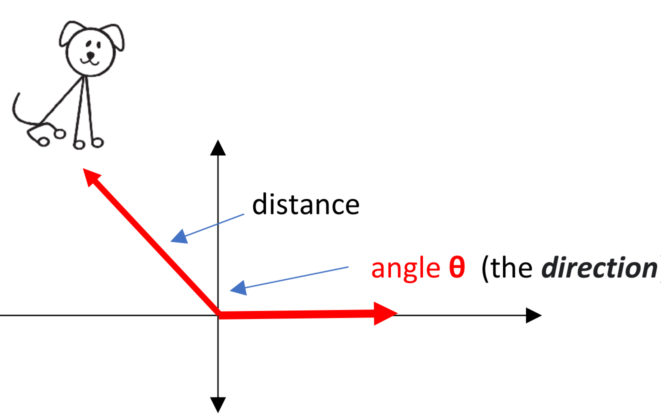
Luckily, we have radar. We locate our radar device at the origin and sweep the **radar signal** for one rotation counterclockwise. The angle **θ** varies from 0 to 2π radians (or from 0 to 360 degrees). There are an infinite number of angles!



As we rotate, our *signal* eventually hits our puppy (who hopefully is not moving – we are keeping this example quite SIMPLE – on purpose!) and bounces back to the origin.

We know how fast the *signal* travels. So, the **time** it takes to bounce back to the origin indicates the ***distance*** - how far away our puppy has roamed from the origin.

The angle **θ** indicates the ***direction*** from the origin; it 'points to' our puppy.



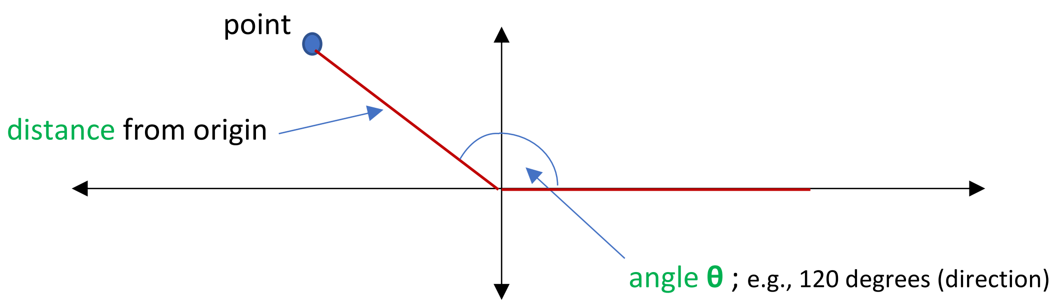
Polar coordinates

We now introduce a second way to determine the position (a.k.a. the location) of a point in a Cartesian plane. We have already discussed the traditional (X,Y) coordinate pair; the 'rectangular' method which requires *two numbers* to locate a point.

We now look at Polar Coordinates

With Polar coordinates we locate a point with *two **numbers*** (sound familiar?).

1. ***Distance*** from to the origin to the point
2. ***Angle* θ** (the measurement of **θ**) formed by the positive X-axis and the point. This is the direction.



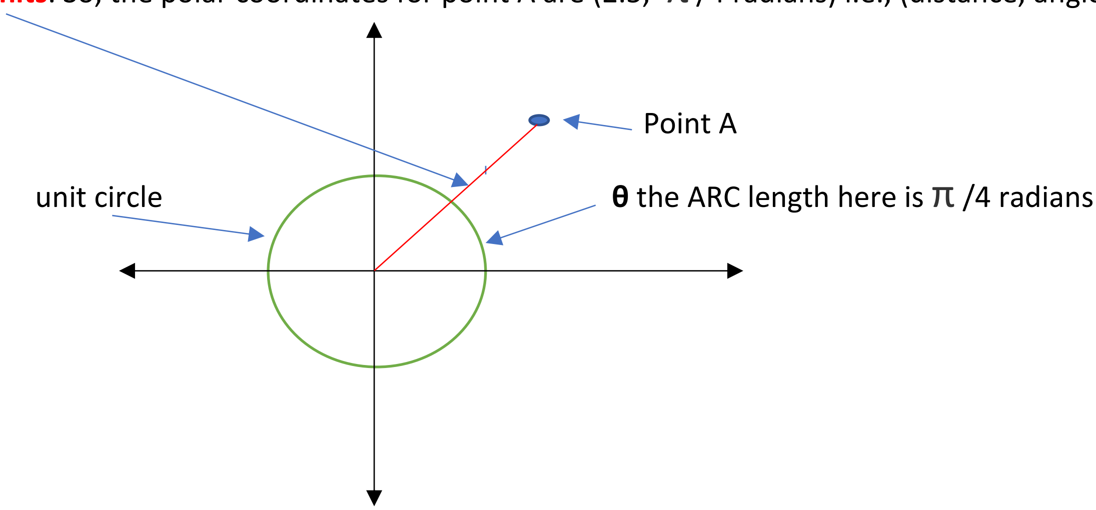
Example

Determining the Polar coordinates of point A.

Construct *the line segment* from the origin to point A. The origin is the 'reference point.' The origin – when using Polar coordinates - is also call the ***pole*** – thus the term Polar coordinates.

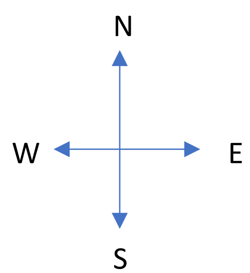
We use our protractor to measure the angle **θ**. For this example (diagram below), we pick the very friendly angle 45 degrees or π /4 radians.

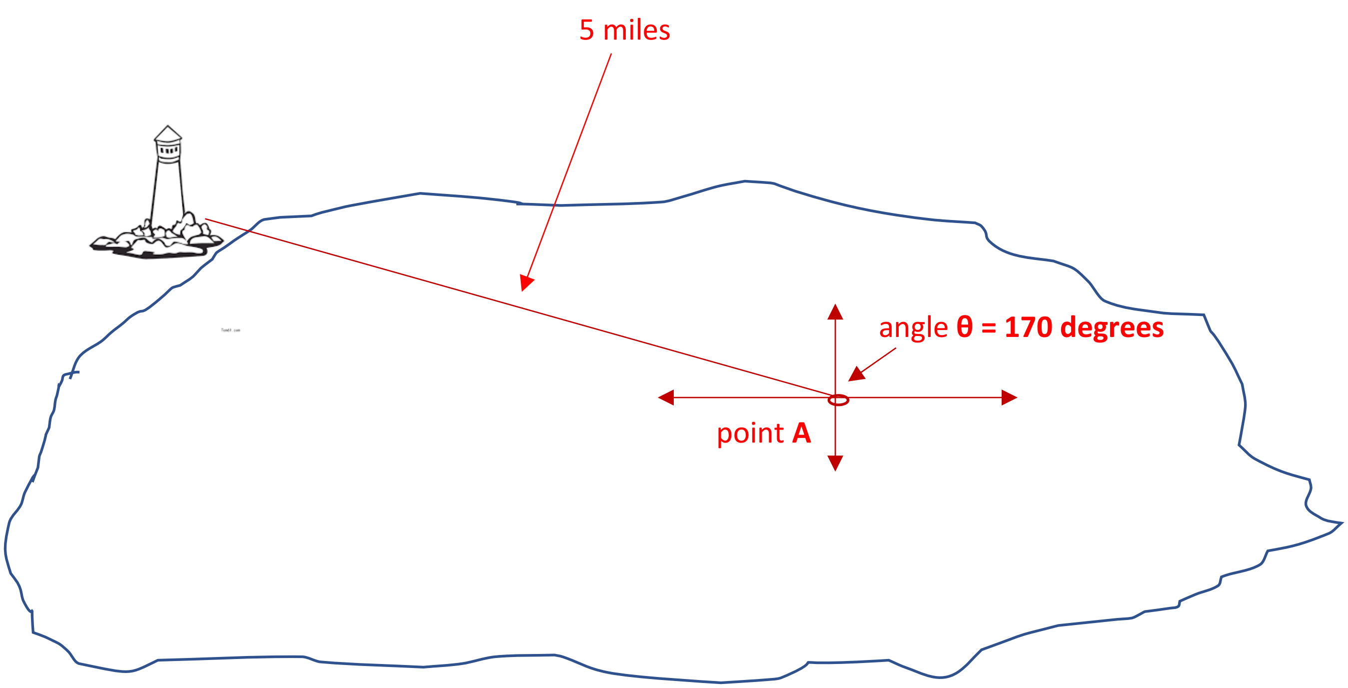
We use a ruler to measure the *line segment* **-** the distance from the origin. Here, we measure 2.5 **units**. So, the polar coordinates for point A are (2.5, π /4 radians) i.e., (distance, angle)



Example

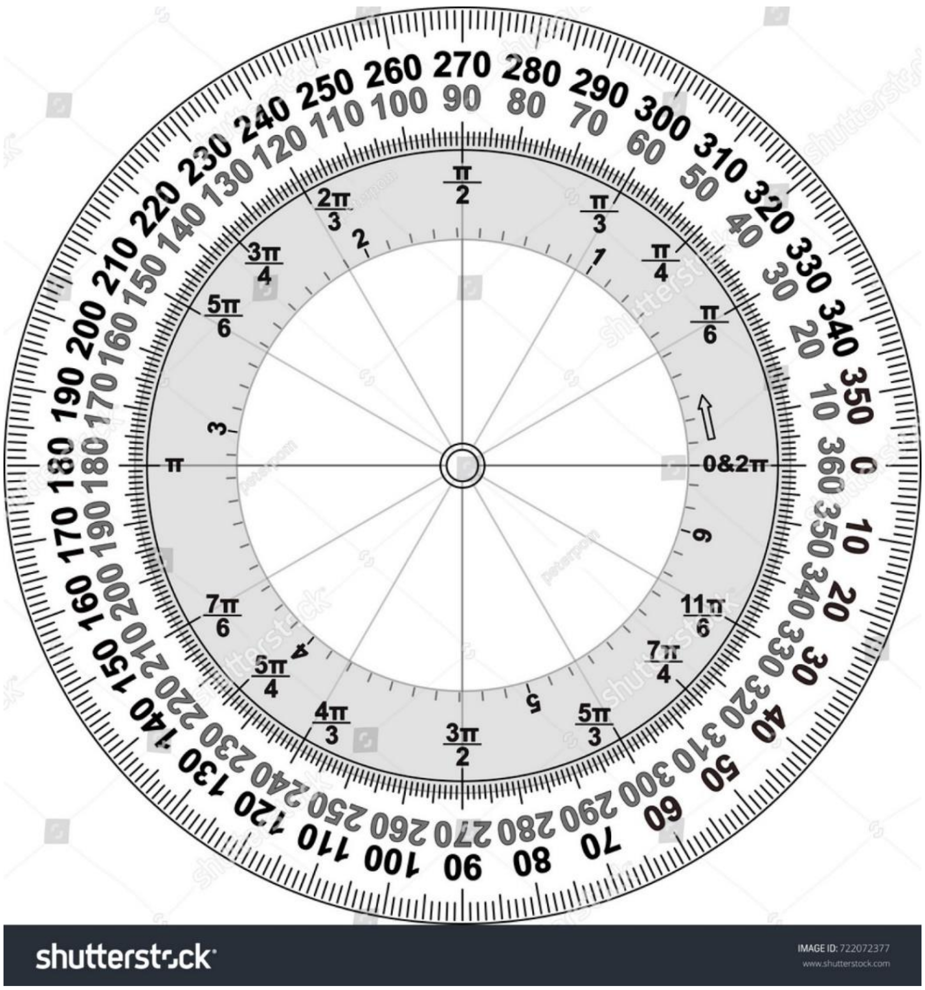
Its dark – nighttime. We can barely see. We are in a boat and lost on a huge **lake**. It is cloudy so we cannot navigate 'by the stars.' There is one light house on the shore, but its bulb burnt out! and we cannot see it- what else could go wrong! Lucky for us, we do have a map of the lake and RADAR. Our RADAR gives us the information below in RED. We use our location - Point A - as the pole (origin). The Polar coordinates of the lighthouse are (5 , 170 degrees). Note: our ***unit*** here is measured in miles.





Comparing XY and Polar Coordinates

Here is our Protractor with degrees and radians.

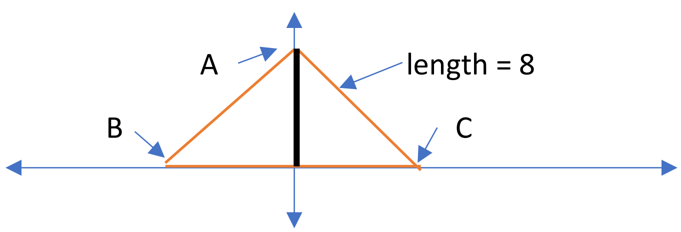


The table below compares some (X,Y) and Polar coordinate pairs.

X-Y Coordinates	Polar coordinates where angle is degrees	Polar coordinates where angle is radians
(8,8)	(8 * √2 , 45)	(8 * √2 , π/4)
(-8, 8)	(8 * √2 , 135)	(8 * √2 , 3π/4)
(-8,-8)	(8 * √2 , 225)	(8 * √2 , 5π/4)
(8,-8)	(8 * √2 , 315)	(8 * √2 , 7π/4)
(8,0)	(8, 0)	(8,0)
(0, 8)	(8, 90)	(8 , π/2)
(-8,0)	(8, 180)	(8 , π)
(0,-8)	(8, 270)	(8 , 3π/2)

Here is an example

We construct an equilateral triangle and position it per the following diagram. Each side is 8 ***units.*** We want to determine the coordinates of the triangle's three vertex points A,B and C.



C and B are easy! They are located at XY-coordinates (4,0) and (-4, 0) respectively. We now determine the **altitude** of the triangle - the **BLACK** segment - via the Pythagorean theorem

$8^2 = \text{altitude}^2 + 4^2$

$64 = \text{altitude}^2 + 16$

$48 = \text{altitude}^2$

$\sqrt{48} = \text{altitude}$

$\sqrt{(16*3)} = \text{altitude}$

$4\sqrt{3} = \text{altitude}$

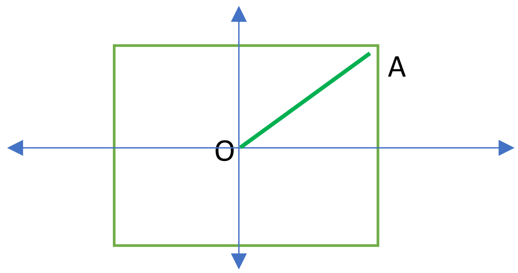
Therefore, A is located at XY-coordinates (4√3 ,0).

Here are our triangle's three vertex points.

Point	***XY-coordinates***	***Polar coordinates***
A	(0, 4√3)	(4√3 , 90 degrees)
B	(-4, 0)	(4, 180 degrees)
C	(4, 0)	(4, 0 degrees)

Example

We construct a square of length 8 and **center** it at the origin - point O. Let c represent the length of segment AB



Pythagorean theorem

$c^2 = 4^2 + 4^2 = 2*16$

$c = \sqrt{2*16} = 4\sqrt{2}$

Our square four vertex points coordinates are. The Polar coordinate's angles are easy to calculate sine the square is centered about the origin

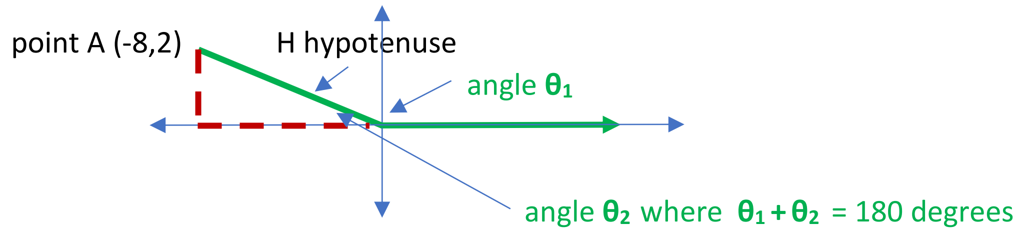
XY	Polar
(4, 4)	(4√2, 45 degrees)
(-4, 4)	(4√2, 135 degrees)
(-4, -4)	(4√2, 225 degrees)
(4, -4)	(4√2, 315 degrees)

All four Polar coordinates have distance of 4√2; vertex points are equidistant from the origin.

Notice: the sign of the XY coordinates determine the direction; for x {east or west} and for y {north or south}. For Polar coordinates, the direction is always positive - if we keep the measure of the angle: [0 to 360 degrees) or [0 to 2 π radians); this range is enough to locate any point in the plane.

Example

Goal: determine angle θ_1



Determine H via the Pythagorean theorem for the right triangle in our diagram

$H^2 = 8^2 + 2^2$

$H^2 = 64 + 4 = 68 = 4 * 17$

$H = 2 * \sqrt{17}$

Now we determine angle θ_2 - we do NOT use a Protractor we choose to use a calculator instead.

We zero in on the right triangle in our diagram which includes angle θ_2 (acute angle). Instead of θ_2 we initially look at sine(θ_2).

sine(θ_2) = opposite / hypotenuse
sine(θ_2) = 2 / (2 * √17)
sine(θ_2) = 1 / √17 ← we now remove square root from the denominator
sine(θ_2) = (1 / √17) * (√17 / √17) ← by multiplying by 1 in the form √17 / √17
sine(θ_2) = √17 / 17

sine(θ_2) ≈ 0.24253562503633297

We now use a calculator to find out what angle has sine(θ_2) = √17 / 17; the calculator give us the value 14.03624346792647858 degrees Via my calculator:

arcSin(0.24253562503633297) ≈ 14.0362434679 degrees

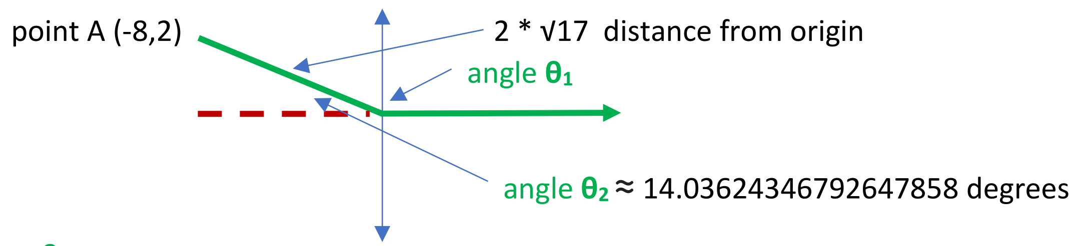
We can 'check' the above line via

sine(14.0362434679 degrees) ≈ 0.24253562503633297

Here is our original diagram. Here we include our measurements and calculations from above.

We measured

- distance from origin to point A = $2 * \sqrt{17}$
- angle $\theta_2 \approx 14.0362434679$ degrees



We calculate θ_1

$\theta_1 + \theta_2 = 180$ degrees

$\theta_1 = 180 - \theta_2$

$\theta_1 \approx 180 - 14.0362434679$

$\theta_1 \approx 165.9637565$ degrees

Finally, we conclude that the Polar coordinates of point A are ($2 * \sqrt{17}$, 165.9637565 degrees)

Time-out! I used the 'arcSin' function on my calculator.

We now investigate arcSin via an example with the nice angle 60 degrees

On a calculator we can calculate the sine of an angle, so for example I ask my ***calculator*** the question:

What is the sine of a 60-degree angle?

and ***it*** calculates 0.86602540378443864676372317075294.

sine(60 degrees) ≈ 0.86602540378443864676372317075294.

I can also ask my ***calculator*** ***the question backwards***

What angle has the sine ***0.86602540378443864676372317075294*** ?

and ***it*** calculates two angles 60 degrees and 120 degrees which **both** have the ***above value*** for its sine. Since we know that we are in the first quadrant (60 degrees in the diagram), we determine that our ***angle*** is 60 degrees.

To ask ***the question backwards*** we use what is called the calculators inverse sine function.

For the word inverse see Merriam-Webster

https://www.merriam-webster.com/dictionary/inverse

- Adjective: opposite in order, nature, or effect
- Noun: something of a contrary nature or quality **:** OPPOSITE, REVERSE

There are two common and confusing names given to the inverse sine function

1. arcSin
2. sine^{-1}

The sine^{-1} name is especially confusing.

These are all different!

sine(x^2) => square x and then take evaluate the sine

example in degrees: sine(9^2) = sine(81 degrees) ≈ 0.9876883405951377

$(\text{sine}(x))^2$ => evaluate the sine and the square it

example in degrees: $(\text{sine}(9))^2$ = sine(9) * sine(9) ≈ 0.156434465040230869^2 = 0.0244717418524232139

$(\text{sine}(x))^{-1}$ => evaluate the sine and then raise to -1 power

example in degrees: $(\text{sine}(9))^{-1}$ ≈ $0.156434465040230869^{-1}$ = 1/0.156434465040230869 = 6.39245322149966

These two are the same

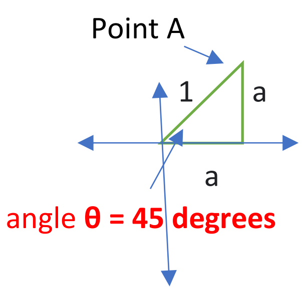
1. $\text{sine}^{-1}(x)$ => evaluate the inverse sine function
2. arcSin(x) => evaluate the inverse sine function

Example in degrees: sine^{-1}(0.7071067811865475) ≈ 45 degrees

This is because sine(45 degrees) = 0.7071067811865475

Also note: 0.7071067811865475 = **√2/2.** Do you remember the following?!

A 45-45-90 (isosceles) right triangle on the **unit** circle



$1^2 = a^2 + a^2$ <- Pythagorean theorem

$1 = 2a^2$

$1/2 = a^2$

$1/\sqrt{2} = a$

$1/\sqrt{2} * \sqrt{2}/\sqrt{2} = a$

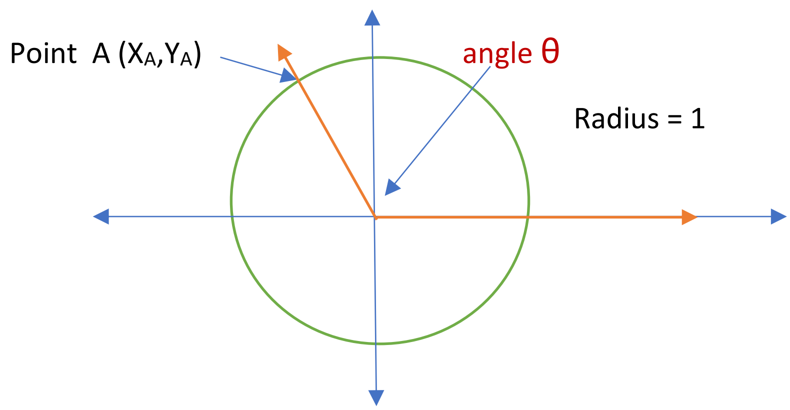
$\sqrt{2}/2 = a$

On the unit circle, point A has XY-coordinates (√2/2, √2/2). Therefore, the sine(**θ)** = sine(45 degrees) = √2/2 -- the Y-coordinate of point A

Earlier we generalized the trigonometric ratios by eliminating an *'underlying' right triangle* and replacing it with XY-coordinates on the unit circle. This allowed us to determine the sine of a 135-degree angle - whereas the *'underlying' right triangle* ratios limited us to angles **θ** where 0 < **θ** < 90

We generalized and 'allowed' our point of intersection to be any point on the unit circle – here we call that point A with coordinates (X_A,Y_A). The point can be **anywhere** on the unit circle. We define our ratios.

- sine(θ) = Y
- cosine(θ) = X
- tangent(θ) = Y/X for X not zero!



Example

We construct a unit circle. Point O is the origin. We construct **the orange dashed line segment** parallel to the x-axis which intersects the circle at two points A and B .

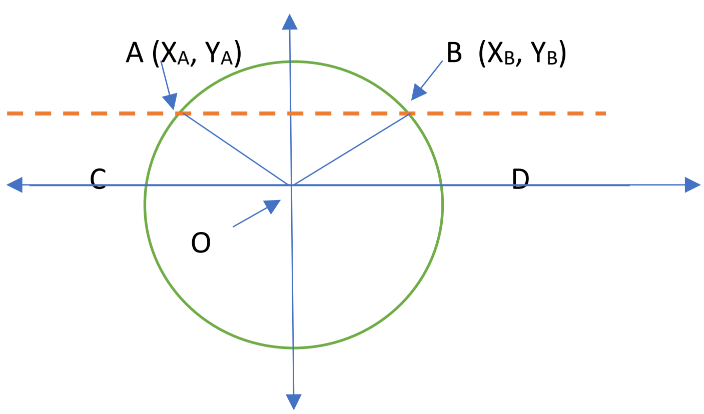
Points A and B have the same y coordinates i.e., $Ya = Y_B$ (they are the same height)

Points A and B have opposite x coordinates i.e., $X_B = - X_A$.

Points C and D are on the x-axis. The following two sines are equal. The following two cosines are opposites.

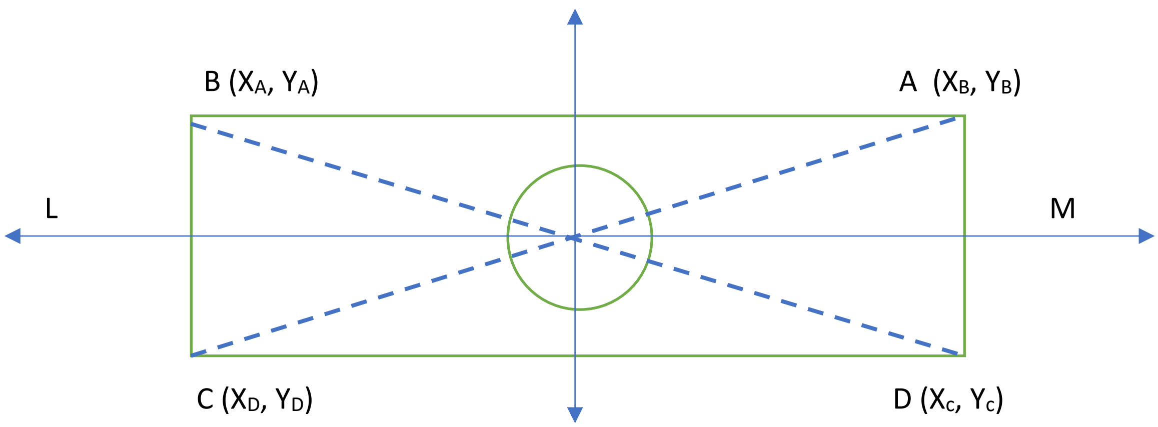
$\text{sine}(\angle DOB) = Y_B$ $\quad$ $\text{cosine}(\angle DOB) = X_B$
$\text{sine}(\angle DOA) = Y_A$ $\quad$ $\text{cosine}(\angle DOA) = X_A$

What would this example look like if we drew our orange line segment parallel to the y-axis?



Example

We construct a unit circle. Point O is the origin. We construct a rectangle centered at O; this means that the rectangle's two diagonals intersect at O. Points L and M are on the x-axis



We look at the tangents of these angles. We measure the angles counterclockwise, e.g., we measure **$270 < \angle MOD < 360$ AND NOT** ~~$0 < \angle MOD < 90$~~ ← ~~clockwise~~

Range in degrees	Tangent	Sign of tangent
$0 < \angle MOA < 90$	tangent($\angle$MOA) = Y_B / X_B	positive
$90 < \angle MOB < 180$	tangent($\angle$MOB) = Y_a / X_a	negative
$180 < \angle MOC < 270$	tangent($\angle$MOC) = Y_D / X_D	positive
$270 < \angle MOD < 360$	tangent($\angle$MOD) = Y_c / X_c	negative

...and because our rectangle is centered at the origin:

$$\text{tangent}(\angle MOA) = -\ \text{tangent}(\angle MOB) = \text{tangent}(\angle MOC) = -\ \text{tangent}(\angle MOD)$$

Suppose we measured $\angle MOA$ and found that it was 25 degrees

Angle in degrees	Tangent via calculator
$\angle MOA = 25$	tangent(25 degrees) = 0.46630765815499859283000619479956
$\angle MOB = 180 - 25 = 155$	tangent(155 degrees) = -0.46630765815499859283000619479956
$\angle MOC = 180 + 25 = 205$	tangent(205 degrees) = 0.46630765815499859283000619479956
$\angle MOD = 360 - 25 = 335$	tangent(335 degrees) = -0.46630765815499859283000619479956

Equations Functions and the Inverse of Tangent

Here are a few example equations with two variables. We use the conventional X and Y as our two variables. For a given X value we can determine the Y value(s). The complete set of X values is called the Domain. The set of all Y values is called the Range

Equation	Domain	Range
$Y = 2*X + 3$	$-\infty < X < \infty$	$-\infty < Y < \infty$
$Y = X^2 + 1$	$-\infty < X < \infty$	$Y >= 1$
$Y = \sqrt{X}$	$X >= 0$	$-\infty < Y < \infty$
$Y = +\sqrt{X}$	$X >= 0$	$0 < Y < \infty$

We categorize an equation as a ***function*** IF for one X in the domain there is one Y value

You can think of the equation as a box where and X enters (input), and a Y is produced (output); here are some examples

for $Y = 2*X + 3$ 4 → [ ] → 11

for $Y = X^2 + 1$ 3 → [ ] → 10

for $Y = X^2 + 1$ -3 → [ ] → 10

for $Y = \sqrt{X}$ 9 → [ ] → +3
→ -3

for $Y = +\sqrt{X}$ 9 → [ ] → +3

Look closely and notice that $Y = \sqrt{X}$ does NOT qualify as a ***function.*** The other **three** equations do. Going forward we limit ourselves here to equations thar are ***functions***

Vertical line test:

> We first graph a given equation in the Cartesian plane. If we can draw one or more vertical lines that intercept the graph in more than ONE point then the equation FAILS the ***function*** test, i.e., it is NOT a ***function***. Try testing our four equations.

Now, we now compare the two ***functions*** $Y = 2*X + 3$ versus $Y = X^2 +1$ and notice a difference. For $X^2 +1$, there are two X values {3 and -3} that both produce the same Y value of 10. This characteristic does NOT apply to $Y = 2*X + 3$.

The function $Y = 2*X + 3$ is said to be 'one to one' - meaning that ***for one X in the domain there is one Y*** *and* **for one Y there is one X**. The function $Y = X^2 +1$ is NOT 'one to one'.

Remember: we say that an equation is a ***function*** IF ***for one X in the domain there is one Y*** value - so $Y = X^2 + 1$ qualifies as a ***function…***BUT it is NOT 'one to one'

Horizontal line test:

> We first graph a given equation in the Cartesian plane. If we can draw one or more horizontal lines that intercept the graph in more than ONE point then the equation FAILS the ***one-to-one*** test, i.e., it is NOT ***one to one***. Try testing our three ***functions***.

The Inverse

We continue to look at our functions

Equation	Domain	Range
$Y = 2*X + 3$	$-\infty < X < \infty$	$-\infty < Y < \infty$
$Y = X^2 + 1$	$-\infty < X < \infty$	$Y >= 1$

We switch the X and Y variables and solve for the 'new' Y! What!? Take a look!

$Y = 2*X + 3$	$Y = X^2 + 1$	
$X = 2 * Y + 3$	$X = Y^2 + 1$	← switch variables
$X - 3 = 2 * Y$ `	$X - 1 = Y^2$	← solve for Y
$(X-3) / 2 = Y$	$\sqrt{(X - 1)} = Y$	
$Y = (X-3) / 2$	$Y = \sqrt{(X - 1)}$	

Finally, we include the domain and range - which have switched

Equation	Inverse	Inverse Domain	Range Domain
$Y = 2*X + 3$	$Y = (X-3) / 2$	$-\infty < X < \infty$	$-\infty < Y < \infty$
$Y = X^2 + 1$	$Y = \sqrt{(X - 1)}$	$X >= 1$	$-\infty < Y < \infty$

We have created the 'inverse' equations for our two functions.

For $Y = 2*X + 3$ (5, 13) and (-10, -17) are two solutions.
For $Y = (X-3) / 2$ (13, 5) and (-17, -10) are the two corresponding 'inverse' solutions!

For $Y = 2*X + 3$ 4 → [ ] → 11

For $Y = (X-3) / 2$ 11 → [ ] → 4

For $Y = X^2 + 1$ (2, 5) and (-2, 5) are two solutions.
For $Y = \sqrt{(X - 1)}$ (5, 2) (5,-2) are two solutions OH NO! - our **inverse** is NOT a ***function***

Note $Y = +\sqrt{(X - 1)}$ is a ***function***

BUT! If we change $Y = \sqrt{(X - 1)}$ to $Y = +\sqrt{(X - 1)}$ we see that (2, 5) has its inverse pair (5,2)
BUT (-2,5) does NOT! So, we have a ***function,*** but it is **not** the **inverse** of $Y = X^2 + 1$

We can generalize: If a ***function*** is ***one to one*** then its inverse is also a ***function***

We can restrict the domain of a function that is NOT ***one to one*** and create a new function that is ***one to one***. We then can create an inverse function

We change (limit) the **domain**

Equation	**Domain**	Range
$Y = X^2 + 1$	**X >= 0**	Y >= 1

Now, we build the inverse as before and use +√ versus √

$Y = X^2 + 1$
X = 2 * Y + 3   switch variable names
$X - 1 = Y^2$ solve for Y
√(X - 1) = Y
Y = +√(X - 1)

Now, we have this

Equation	Inverse	Inverse Domain	Range Domain
$Y = X^2 + 1$	Y = √(X - 1)	X >= 1	**Y >=0**

domain and range switched

Now our ***Dilemma*** is gone:
For $Y = X^2 + 1$ (2, 5)   Our Domain limitation removed (-2,5)
For Y = +√(X - 1)   (5, 2)   This is the **inverse** and is a ***function***!

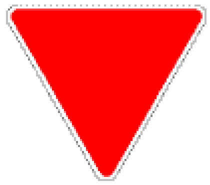
Trig Tangent and the Inverse

We discuss this topic in Volume two.

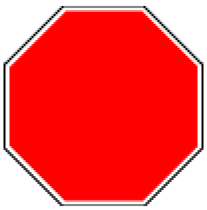
Regular Polygons

A *polygon* is a 'many' (3 or more) sided geometric figure. We are all familiar with examples - a triangle, a square, a rectangle, a trapezoid, a stop sign, etc.

YIELD!



STOP!



A *regular polygon* is a polygon whose sides are all the same length AND whose angles all have the same measure. Examples include equilateral triangles, squares and stop signs. Traffic signs often are regular polygons - this make it easier for drivers to distinguish among multiple signs.

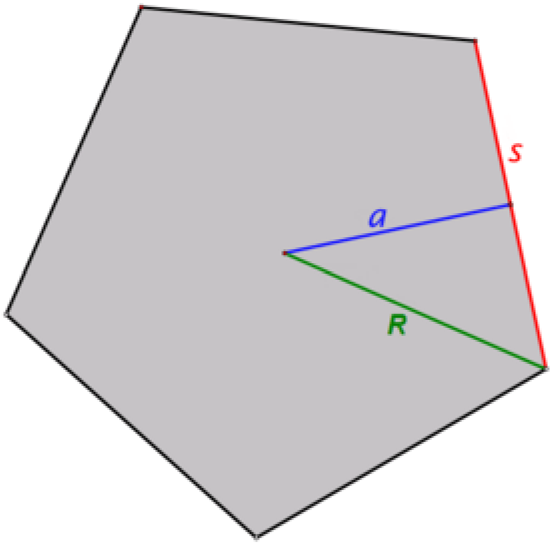
Regular polygons are symmetrical around a single *center point* - like a circle

The *circumradius R* is the distance from the center of a regular polygon to one of the vertices - again like a circle. We can say one because any side will provide the same distance - the shape is symmetrical around the center point

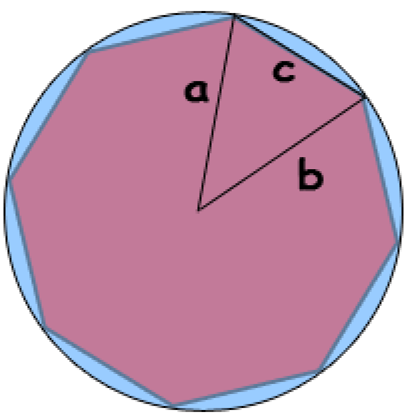
The *apothem* (sometimes abbreviated as **apo**) of a regular polygon is a line segment from the center to the midpoint of one of its sides. Equivalently, it is the line drawn from the center of the polygon that is perpendicular to one of its sides.

The apothems in a regular polygon are all the same length - again! the shape is symmetrical around the center point.

In the following diagram we see the side S, the apothem, and the circumradius R



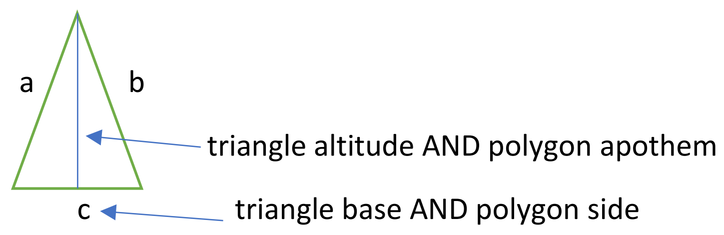
Because of their symmetry a regular polygon is 'nicely' inscribed within a circle. The segments a and b are two circumradii used here to construct an isosceles triangle where a = b. C is one side of the polygon



We determine the area of the triangle.

Formula: Area of triangle = altitude * base / 2

Here is a copy of the isosceles triangle from above. I include an apothem. This apothem is the triangles altitude. The triangles base is one side of the polygon with length c



Area of triangle = base * altitude / 2
= side * apothem /2

Some formulas for a *regular polygon* of **N** sides (**N** >= 3)

Perimeter = **N * side**

Area = N * area of our triangle
= **N * side** * apothem /2
= **perimeter** * apothem /2

Volume one epilogue

In volume one (what you are now reading!) we have reviewed=ed a bunch of “High school’ math level math concepts. We included a bit of Geometry and Trigonometry. We reviewed the XY-Coordinates and introduced the alternative Polar coordinates.

In Volume two we will switch to a software ‘programming mode’ and examine some coding examples emphasizing the two coordinate systems.

The end